WAEC Math

Complete Student Guide

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Chapter 1: Number and Numeration

1.1 Fractions and Decimals

Fractions and decimals are fundamental concepts in arithmetic. Operations with fractions include addition, subtraction, multiplication, and division. Fractions can also be simplified to their lowest terms by determining the greatest common factor (GCF) of the numerator and denominator.

Decimals are numbers written in a base-10 system and can be converted to fractions and vice versa.

Key Concepts:

- Least Common Multiple (LCM): The smallest number that is a multiple of two or more numbers.
- Highest Common Factor (HCF): The largest number that divides two or more numbers without leaving a remainder.
- Factorization: Breaking a number into its prime factors.

Example 1: Addition of Fractions

$$\begin{aligned}
\text{Simplify: } & \frac{3}{4} + \frac{5}{6}. \\
& \textbf{Solution:} \\
\text{Find the LCM of 4 and 6: } & \text{LCM} = 12. \\
\text{Convert the fractions: } & \frac{3}{4} = \frac{9}{12}, \quad \frac{5}{6} = \frac{10}{12}. \\
\text{Add: } & \frac{9}{12} + \frac{10}{12} = \frac{19}{12}.
\end{aligned}$$

Example 2: Multiplication of Decimals

$$\begin{aligned}
\text{Simplify: } & 1.25 \times 0.4. \\
& \textbf{Solution:} \\
\text{Multiply as integers: } & 125 \times 4 = 500. \\
\text{Adjust decimal places: } & 1.25 \text{ has 2 decimal places, and 0.4 has 1.} \\
\text{Final answer: } & 0.500 \text{ or } 0.5.
\end{aligned}$$

Example 3: Finding the HCF

$$\begin{aligned}
\text{Find the HCF of 48 and 180.} & \\
\textbf{Solution:} & \\
\text{Step 1: Perform prime factorization.} & \\
& 48 = 2^4 \times 3, \quad 180 = 2^2 \times 3^2 \times 5. \\
\text{Step 2: Identify common factors.} & \\
\text{The common factors are } 2^2 \text{ and } 3. & \\
& \text{HCF: } 2^2 \times 3 = 12.
\end{aligned}$$

Example 4: Factorization of a Number

Factorize 84 into its prime factors.

Solution:

Divide successively by prime numbers:

$$84 \div 2 = 42, \quad 42 \div 2 = 21, \quad 21 \div 3 = 7.$$

Prime factorization: $84 = 2^2 \times 3 \times 7$.

1.2 Indices and Standard Form

The laws of indices are used to simplify expressions involving powers. They include:

- $a^m \cdot a^n = a^{m+n}$ (Product Rule)
- $\frac{a^m}{a^n} = a^{m-n}$ (Quotient Rule)
- $(a^m)^n = a^{mn}$ (Power of a Power)
- $a^0 = 1$ for $a \neq 0$ (Zero Exponent Rule)
- $a^1 = a$ (Identity Rule)
- $a^{-n} = \frac{1}{a^n}$ (Negative Exponent Rule)
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ (Fractional Exponent Rule)

Example 1: Simplifying Powers

$$\text{Simplify: } (3^2 \cdot 3^3) \div 3.$$

Solution:

$$\text{Step 1: Apply product rule: } 3^2 \cdot 3^3 = 3^{2+3} = 3^5.$$

$$\text{Step 2: Divide: } \frac{3^5}{3} = 3^{5-1} = 3^4 = 81.$$

Example 2: Converting to Standard Form

$$\text{Express 0.00045 in standard form.}$$

Solution:

$$\text{Write as a product: } 0.00045 = 4.5 \times 10^{-4}.$$

Example 3: Negative and Fractional Indices

$$\text{Simplify: } 16^{-\frac{3}{4}}.$$

Solution:

$$\text{Step 1: Apply fractional exponent rule: } 16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3.$$

$$\text{Step 2: Simplify: } 16^{\frac{1}{4}} = 2, \text{ so } 2^3 = 8.$$

$$\text{Step 3: Apply negative exponent rule: } 16^{-\frac{3}{4}} = \frac{1}{8}.$$

1.3 Number Bases

Number bases represent the numerical system in which values are expressed. The base indicates the number of unique digits, including zero, that a numeral system uses. Common bases include:

- Base 10 (Decimal): Standard numeric system.
- Base 2 (Binary): Used in computer systems.
- Base 16 (Hexadecimal): Common in programming and computing.

Key Concepts:

- Converting numbers between bases involves successive division by the new base and noting remainders.
- To perform operations on numbers in any base first convert them to base 10, perform the operation and then convert the answer back to original base.

Example 1: Conversion from Decimal to Binary

$$\begin{aligned} &\text{Convert 18 from base 10 to base 2.} \\ &\textbf{Solution:} \\ &\text{Step 1: Divide by 2 and record remainders:} \\ &\qquad 18 \div 2 = 9 \text{ remainder } 0, \\ &\qquad 9 \div 2 = 4 \text{ remainder } 1, \\ &\qquad 4 \div 2 = 2 \text{ remainder } 0, \\ &\qquad 2 \div 2 = 1 \text{ remainder } 0, \\ &\qquad 1 \div 2 = 0 \text{ remainder } 1. \\ &\text{Step 2: Write remainders in reverse order: } 10010_2. \end{aligned}$$

Example 2: Conversion from Binary to Decimal

$$\begin{gathered} \text{Convert } 1011_2 \text{ to base 10.} \\ \textbf{Solution:} \\ \text{Apply place values:} \\ (1 \cdot 2^3) + (0 \cdot 2^2) + (1 \cdot 2^1) + (1 \cdot 2^0) = 8 + 0 + 2 + 1 = 11. \end{gathered}$$

Example 3: Multiplication in Base 3

$$\begin{aligned} &\text{Simplify } 21_3 \cdot 2_3 \text{ in base 3.} \\ &\textbf{Solution:} \\ \text{Convert each to decimal: } & 21_3 = (2 \cdot 3^1) + (1 \cdot 3^0) = 6 + 1 = 7. \\ & 2_3 = 2. \\ \text{Multiply in decimal: } & 7 \cdot 2 = 14. \\ \text{Convert 14 to base 3: } & 14 \div 3 = 4 \text{ remainder } 2, \quad 4 \div 3 = 1 \text{ remainder } 1. \\ \text{Result: } & 112_3. \end{aligned}$$

Example 4: Squaring in Base 2

Simplify: $(11_{\text{two}})^2$.

Solution:

Convert to decimal: $11_{\text{two}} = 3$.

Square in decimal: $3^2 = 9$.

Convert back to binary: $9 \div 2 = 4$ remainder $1, \quad 4 \div 2 = 2$ remainder $0, \quad 2 \div 2 = 1$ remainder $0, \quad 1$

Result: 1001_{two}.

1.4 Logarithms

Logarithms are the inverses of exponents. For $a^x = b$, $\log_a b = x$.

Laws of Logarithms:

- $\log(ab) = \log a + \log b$ (Product Rule)
- $\log\left(\frac{a}{b}\right) = \log a - \log b$ (Quotient Rule)
- $\log(a^n) = n \log a$ (Power Rule)

Example 1: Simplify $\log\left(\frac{28}{3}\right)$ given $\log 2 = x$, $\log 3 = y$, and $\log 7 = z$.
Solution:
Step 1: Apply the logarithm property for division:

$$\log\left(\frac{28}{3}\right) = \log 28 - \log 3.$$

Step 2: Break down 28 into factors:

$$\log 28 = \log(4 \cdot 7) = \log 4 + \log 7.$$

Step 3: Substitute and simplify:

$$\log\left(\frac{28}{3}\right) = (\log 4 + \log 7) - \log 3.$$

Step 4: Replace $\log 4$ with $2 \log 2$:

$$\log\left(\frac{28}{3}\right) = (2 \log 2 + \log 7) - \log 3.$$

Therefore, $\log\left(\frac{28}{3}\right) = 2x + z - y$.

Example 2: Evaluate $x \log_2 8$ given $\log_x 64 = 3$.
Solution:
Step 1: Express $\log_x 64 = 3$ in exponential form:

$$x^3 = 64 \implies x = 4.$$

Step 2: Evaluate $\log_2 8$:

$$\log_2 8 = \log_2(2^3) = 3.$$

Step 3: Multiply x by $\log_2 8$:

$$x \log_2 8 = 4 \times 3 = 12.$$

Therefore, $x \log_2 8 = 12$.

Example 3: Solve $\log_3 27 = 2x + 1$ to find the value of x.
Solution:
Step 1: Simplify $\log_3 27$:

$$\log_3 27 = \log_3(3^3) = 3.$$

Step 2: Substitute into the equation:

$$3 = 2x + 1.$$

Step 3: Solve for x:

$$2x = 3 - 1 = 2.$$

$$x = \frac{2}{2} = 1.$$

Therefore, $x = 1$.

1.5 Ratio and Proportion, Percentages

Ratios compare two quantities.

Two quantities are said to be **proportional** if one quantity is a constant multiple of the other. In direct proportionality, as one quantity increases, the other increases at the same rate, and this relationship can be expressed as $y \propto x$ or $y = kx$, where k is the constant of proportionality. Conversely, two quantities are **inversely proportional** if one quantity increases while the other decreases such that their product remains constant. This relationship is expressed as $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$, where k is the constant of proportionality.

Example 1: Solve the ratio $\frac{2}{5} = \frac{x}{15}$.
Step 1: Cross multiply the terms in the equation:

$$2 \cdot 15 = 5 \cdot x.$$

Step 2: Simplify the multiplication:

$$30 = 5x.$$

Step 3: Solve for x:

$$x = \frac{30}{5} = 6.$$

Therefore, $x = 6$.

Example 2: Three boys shared D 10,500.00 in the ratio 6:7:8. Find the largest share.
Step 1: Find the total ratio:

$$6 + 7 + 8 = 21.$$

Step 2: Divide the total amount by the total ratio to find the value of one part:

$$\text{Value of one part} = \frac{10,500}{21} = 500.$$

Step 3: Find the largest share:

$$\text{Largest share} = 8 \times 500 = 4,000.$$

Therefore, the largest share is D 4,000.

Example 3: Given that $P \propto \frac{1}{\sqrt{r}}$ and $P = 3$ when $r = 16$, find the value of r when $P = \frac{3}{2}$.
Solution:
Step 1: Express P in terms of r using the proportionality constant k:

$$P = \frac{k}{\sqrt{r}}.$$

Step 2: Substitute $P = 3$ and $r = 16$ to find k:

$$3 = \frac{k}{\sqrt{16}}.$$

$$3 = \frac{k}{4}.$$

$$k = 3 \times 4 = 12.$$

Step 3: Use $k = 12$ and $P = \frac{3}{2}$ to find r:

$$\frac{3}{2} = \frac{12}{\sqrt{r}}.$$

$$\sqrt{r} = \frac{12}{\frac{3}{2}}.$$

$$\sqrt{r} = \frac{12 \times 2}{3} = 8.$$

Step 4: Square both sides to find r:

$$r = 8^2 = 64.$$

Therefore, the value of r when $P = \frac{3}{2}$ is 64.

Example 4: Ladi sold a car for N84,000 at a loss of 4%. How much did Ladi buy the car?
Solution:
Step 1: Recall the formula for selling price with a loss:

$$\text{Selling Price (SP)} = \text{Cost Price (CP)} - \text{Loss}.$$

Step 2: Express the loss in terms of the cost price:

$$\text{Loss} = \frac{\text{Loss Percentage} \times \text{Cost Price}}{100}.$$

Step 3: Substitute into the selling price formula:

$$\text{SP} = \text{CP} - \frac{\text{Loss Percentage} \times \text{CP}}{100}.$$

$$84,000 = \text{CP} - \frac{4 \times \text{CP}}{100}.$$

Step 4: Simplify the equation:

$$84,000 = \text{CP}\left(1 - \frac{4}{100}\right).$$

$$84,000 = \text{CP} \times 0.96.$$

Step 5: Solve for the cost price (CP):

$$\text{CP} = \frac{84,000}{0.96}.$$

$$\text{CP} = 87,360.$$

Therefore, Ladi bought the car for N87,360.

1.6 Surds

Surds are irrational numbers that cannot be simplified into exact decimals or fractions but can be expressed using a radical symbol (e.g., $\sqrt{2}$). Surds are often simplified or rationalized in expressions to make calculations easier.

Key Concepts:

- Simplification of surds involves finding factors that are perfect squares and simplifying them.
- Rationalization involves removing surds from the denominator by multiplying the numerator and denominator by an appropriate value.
- The conjugate of a binomial with surds is used to rationalize denominators with two terms.

Example 1: Simplify $\sqrt{50}$.
Solution:
Step 1: Find factors of 50 where one is a perfect square:

$$50 = 25 \times 2.$$

Step 2: Take the square root of the perfect square:

$$\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}.$$

Therefore, $\sqrt{50} = 5\sqrt{2}$.

Example 2: Rationalize $\frac{1}{\sqrt{3}}$.
Solution:
Step 1: Multiply numerator and denominator by $\sqrt{3}$:

$$\frac{1}{\sqrt{3}} = \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}.$$

Step 2: Simplify:

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Therefore, the rationalized form is $\frac{\sqrt{3}}{3}$.

Example 3: Simplify and rationalize $\frac{\sqrt{8}}{2+\sqrt{2}}$.
Solution:
Step 1: Simplify the numerator:

$$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}.$$

Step 2: Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{\sqrt{8}}{2 + \sqrt{2}} = \frac{2\sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}}.$$

Step 3: Expand the numerator and denominator:

$$\text{Numerator: } 2\sqrt{2} \cdot (2 - \sqrt{2}) = 4\sqrt{2} - 2\sqrt{4} = 4\sqrt{2} - 4.$$

$$\text{Denominator: } (2 + \sqrt{2})(2 - \sqrt{2}) = 4 - 2 = 2.$$

Step 4: Simplify:

$$\frac{\sqrt{8}}{2 + \sqrt{2}} = \frac{4\sqrt{2} - 4}{2} = 2\sqrt{2} - 2.$$

Therefore, the simplified and rationalized form is $2\sqrt{2} - 2$.

1.7 Sets

Set theory is a fundamental branch of mathematics that deals with collections of objects, called sets. These objects are called elements or members of the set.

Key Concepts:

- **Union of Sets:** The union of two sets A and B, denoted by $A \cup B$, is the set of elements that are in A, in B, or in both.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

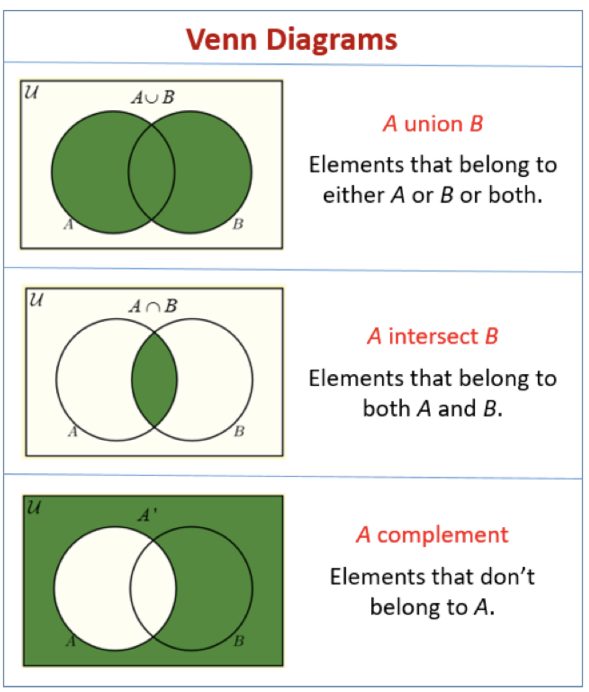
- **Intersection of Sets:** The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements that are in both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- **Complement of a Set:** The complement of a set A, denoted by A', is the set of elements not in A but in the universal set U.

$$A' = \{x \in U \mid x \notin A\}.$$

- **Universal Set:** The universal set, denoted by U, is the set that contains all elements under consideration in a particular context. All subsets are defined within this universal set.
- **Cardinality of a Set:** The cardinality of a set A, denoted by $|A|$, is the number of elements in the set.
- **Venn Diagrams:** Venn diagrams are visual representations of sets and their relationships, such as union, intersection, and complement.



Example 1: Union of Two Sets
Given $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, find $A \cup B$.

Solution:

Step 1: Combine all unique elements of A and B:

$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

Therefore, $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

Example 2: Intersection of Two Sets
Given $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, find $A \cap B$.

Solution:

Step 1: Identify the common elements in A and B:

$$A \cap B = \{3, 4\}.$$

Therefore, $A \cap B = \{3, 4\}$.

Example 3: Complement of a Set
Given $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{2, 4, 6\}$, find A'.

Solution:

Step 1: List all elements in U that are not in A:

$$A' = \{1, 3, 5, 7\}.$$

Therefore, $A' = \{1, 3, 5, 7\}$.

Example 4: Cardinality of Sets and Venn Diagram Application
In a survey, 20 students like basketball, 15 like football, and 10 like both sports. How many students like at least one of the two sports?

Solution:

Step 1: Use the principle of inclusion-exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Step 2: Substitute the values:

$$|A \cup B| = 20 + 15 - 10 = 25.$$

Therefore, 25 students like at least one of the two sports.

Chapter 2: Algebra

2.1 Algebraic Expressions

An algebraic expression is a mathematical statement that includes numbers, variables, and operations. Operations like addition, subtraction, multiplication, division, and exponentiation can be applied to form such expressions. Understanding the fundamental terms is crucial for simplifying and analyzing algebraic expressions.

Key Concepts:

- **Simplification:** Combining like terms and reducing an expression to its simplest form. Like terms are terms with the same variables raised to the same powers.
- **Factorization:** Rewriting an expression as a product of its factors. Common formulas include:
$$a^2 - b^2 = (a-b)(a+b)$$
$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$
$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$
- **Coefficient:** The numerical factor in a term of an algebraic expression. For example, in $5x^2$, the coefficient is 5.
- **Term:** A single mathematical expression involving numbers, variables, or their product. For example, $3x$, $-7y^2$, and 4 are all terms of expression $3x - 7y^2 + 4$.
- **Degree of a Polynomial:** The highest power of the variable in the expression. For example, the degree of $3x^4 + 2x^2 + 1$ is 4.
- **Domain of a Function:** The set of all possible values of the independent variable for which the function is defined. For example:
  - $f(x) = \frac{1}{x-2}$: Undefined when $x = 2$, so the domain is $x \neq 2$.
  - $f(x) = \log(x)$: Defined when $x > 0$, so the domain is $x > 0$.
  - $f(x) = \sqrt{x}$: Defined when $x \geq 0$, so the domain is $x \geq 0$.
- **Undefined Expressions:** Expressions are undefined when division by zero occurs. For example, $\frac{1}{x-3}$ is undefined at $x = 3$.

Example 1: Expand $(x+2)(x-3)$.

Solution:

Use the distributive property:

$$(x+2)(x-3) = x^2 - 3x + 2x - 6 = x^2 - x - 6.$$

Therefore, the expanded expression is $x^2 - x - 6$.

Example 2: Factorize $xy + xz + wy + wz$.

Solution:

Group terms and factorize:

$$xy + xz + wy + wz = x(y + z) + w(y + z) = (x + w)(y + z).$$

Therefore, the factorized expression is $(x + w)(y + z)$.

Example 3: Find the domain of $f(x) = \ln(x - 5)$.

Solution:

The function $f(x) = \ln(x - 5)$ is defined when argument is positive:

$$x - 5 > 0 \implies x > 5.$$

Therefore, the domain is $x > 5$.

Example 4: Find when $\frac{x-2}{x-1}$ is undefined.

Solution:

The expression $\frac{x-2}{x-1}$ is undefined when the denominator is zero:

$$x - 1 = 0 \implies x = 1.$$

Therefore, the expression is undefined at $x = 1$.

Example 5: Factorize $x^2 - 9$.

Solution:

Recognize the expression as a difference of squares:

$$x^2 - 9 = (x - 3)(x + 3).$$

Therefore, the factorized form is $(x - 3)(x + 3)$.

2.2 Equations and Inequalities

Equations and inequalities are fundamental tools in algebra used to express relationships between variables and solve problems.

Key Concepts:

- **Linear Equations:** Equations of the form $ax + b = 0$, where $a \neq 0$.
- **Simultaneous Equations:** A set of equations with multiple variables solved together to find a common solution.
- **Functional Equations:** Equations involving functions where the goal is to determine the function form, e.g., $f(x + 1) = 2f(x)$.
- **Inequalities:** Statements involving $<$, $>$, $\leq$, or $\geq$ that describe the range of solutions for a variable.

Example 1: Mary has $3.00 more than Ben but $5.00 less than Jane. If Mary has $x, how much do Jane and Ben have altogether?

Solution:

Step 1: Define the amounts Jane and Ben have in terms of Mary:

$$\text{Ben's amount: } x - 3, \quad \text{Jane's amount: } x + 5.$$

Step 2: Add their amounts:

$$\text{Total: } (x - 3) + x + (x + 5).$$

Step 3: Simplify:

$$\text{Total: } 3x + 2.$$

Therefore, Jane and Ben have $\$3x + 2$ altogether.

Example 2: Simplify $\frac{2-18m^2}{1+3m}$.

Solution:

Step 1: Factorize the numerator:

$$2 - 18m^2 = 2(1 - 9m^2).$$

Recognize $1 - 9m^2$ as a difference of squares:

$$1 - 9m^2 = (1 - 3m)(1 + 3m).$$

Thus:

$$2 - 18m^2 = 2(1 - 3m)(1 + 3m).$$

Step 2: Simplify the fraction:

$$\frac{2 - 18m^2}{1 + 3m} = \frac{2(1 - 3m)(1 + 3m)}{1 + 3m}.$$

Cancel the common factor $(1+3m)$ (valid when $1+3m \neq 0$):

$$\frac{2(1-3m)(1+3m)}{1+3m} = 2(1-3m).$$

Step 3: Expand if needed:

$$2(1-3m) = 2-6m.$$

Therefore, the simplified expression is $2-6m$.

Example 3: Solve the linear equation $2x-5=7$.

Solution:

Step 1: Add 5 to both sides:

$$2x-5+5 = 7+5 \implies 2x = 12.$$

Step 2: Divide by 2:

$$x = \frac{12}{2} = 6.$$

Therefore, the solution is $x = 6$.

Example 4: Solve the simultaneous equations:

$$x+y=5, \quad 2x-y=4.$$

Solution:

Step 1: Add the equations to eliminate y:

$$(x+y)+(2x-y) = 5+4 \implies 3x = 9.$$

Step 2: Solve for x:

$$x = \frac{9}{3} = 3.$$

Step 3: Substitute $x=3$ into the first equation:

$$3+y=5 \implies y = 5-3 = 2.$$

Therefore, the solution is $x=3$ and $y=2$.

Example 5: Solve the inequality $2x-3>7$.

Solution:

Step 1: Add 3 to both sides:

$$2x-3+3 > 7+3 \implies 2x > 10.$$

Step 2: Divide by 2:

$$x > \frac{10}{2} \implies x > 5.$$

Therefore, the solution is $x > 5$.

Example 6: A Word Problem for a System of Linear Equations
A school sells tickets for a play. Adult tickets cost \$10, and student tickets cost \$6. If the school sells 50 tickets and collects \$400, how many adult and student tickets were sold?

Solution:

Let x represent the number of adult tickets, and y represent the number of student tickets.
Step 1: Write the equations based on the problem:

$$x + y = 50 \quad \text{(total tickets sold)}.$$

$$10x + 6y = 400 \quad \text{(total revenue)}.$$

Step 2: Solve the system of equations. From the first equation:

$$y = 50 - x.$$

Step 3: Substitute $y = 50 - x$ into the second equation:

$$10x + 6(50 - x) = 400.$$

Step 4: Simplify and solve for x:

$$10x + 300 - 6x = 400 \implies 4x + 300 = 400 \implies 4x = 100 \implies x = 25.$$

Step 5: Substitute $x = 25$ into $y = 50 - x$:

$$y = 50 - 25 = 25.$$

Therefore, 25 adult tickets and 25 student tickets were sold.

Example 7: Solve the functional equation $f(x + 1) = 2f(x)$, given $f(1) = 3$.

Solution:

Step 1: Write the values of $f(x)$ using the functional equation:

$$f(x + 1) = 2f(x).$$

Step 2: Find $f(2)$ using $f(1) = 3$:

$$f(2) = 2f(1) = 2 \cdot 3 = 6.$$

Step 3: Find $f(3)$ using $f(2) = 6$:

$$f(3) = 2f(2) = 2 \cdot 6 = 12.$$

Step 4: Generalize the pattern:

$$f(x + 1) = 2f(x) \implies f(x) = 3 \cdot 2^{x-1}.$$

Therefore, the solution to the functional equation is:

$$f(x) = 3 \cdot 2^{x-1}.$$

2.3 Quadratic Equations and Inequalities

Quadratic equations and inequalities involve a variable raised to the second power. They are fundamental in algebra and appear in various real-world applications.

Key Concepts:

- **General Form of a Quadratic Equation:**

$$ax^2 + bx + c = 0, \quad a \neq 0$$

  where a, b, and c are constants.

- **Quadratic Formula:** Used to solve quadratic equations:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- **Ways to Solve Quadratic Equations:**
  - By factorization.
  - By completing the square.
  - By using the quadratic formula.

- **Discriminant:** The discriminant $\Delta = b^2 - 4ac$ determines the nature of the roots:
  - If $\Delta > 0$, there are two distinct real roots.
  - If $\Delta = 0$, there is one real root (a repeated root).
  - If $\Delta < 0$, there are no real roots (two complex roots).

- **Graph of a Quadratic Function:** The graph of $y = ax^2 + bx + c$ is a parabola:
  - If $a > 0$, the parabola opens upwards.
  - If $a < 0$, the parabola opens downwards.

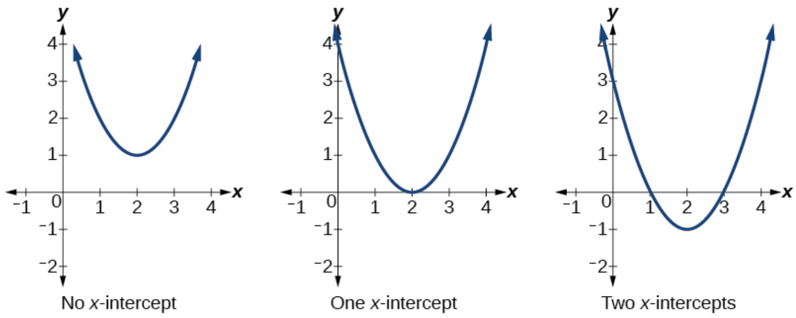


Figure 1: Three cases for quadratic graph, dependin on descriminant.

  The vertex is given by:

$$x = -\frac{b}{2a}.$$

- **Vieta's Formulas for Roots:**

- Sum of roots: $r_1 + r_2 = -\frac{b}{a}$.
- Product of roots: $r_1 \cdot r_2 = \frac{c}{a}$.

Example 1: Solve $x^2 - 5x + 6 = 0$ by factorization.

Solution:

Step 1: Factorize the quadratic equation:

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

Step 2: Solve for x:

$$x - 2 = 0 \implies x = 2, \quad x - 3 = 0 \implies x = 3.$$

Therefore, the solutions are $x = 2$ and $x = 3$.

Example 2: Solve $x^2 + 6x + 5 = 0$ by completing the square.

Solution:

Step 1: Rewrite the equation:

$$x^2 + 6x = -5.$$

Step 2: Complete the square:

$$x^2 + 6x + 9 = -5 + 9 \implies (x + 3)^2 = 4.$$

Step 3: Solve for x:

$$x + 3 = \pm\sqrt{4} \implies x + 3 = 2 \quad \text{or} \quad x + 3 = -2.$$

Step 4: Simplify:

$$x = -1 \quad \text{or} \quad x = -5.$$

Therefore, the solutions are $x = -1$ and $x = -5$.

Example 3: Solve $2x^2 - 3x + 1 = 0$ using the quadratic formula.

Solution:

Identify the coefficients:

$$a = 2, \quad b = -3, \quad c = 1.$$

Use the quadratic formula:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)}.$$

Simplify:

$$x = \frac{3 \pm \sqrt{9 - 8}}{4} = \frac{3 \pm 1}{4}.$$

Solve for x:

$$x = \frac{3 + 1}{4} = 1 \quad \text{or} \quad x = \frac{3 - 1}{4} = \frac{1}{2}.$$

Therefore, the solutions are $x = 1$ and $x = \frac{1}{2}$.

Example 4: Find the nature of the roots for $x^2 + 4x + 5 = 0$.

Solution:

Step 1: Calculate the discriminant:

$$\Delta = b^2 - 4ac = 4^2 - 4(1)(5) = 16 - 20 = -4.$$

Step 2: Interpret the result:

$$\Delta < 0 \implies \text{the equation has two complex roots.}$$

Therefore, the roots are complex.

Example 5: Write a quadratic equation whose roots are $-\frac{1}{2}$ and 3.

Solution:

Step 1: Use Vieta's formulas for roots:

$$\text{Sum of roots: } r_1 + r_2 = -\frac{b}{a}.$$

$$\text{Product of roots: } r_1 \cdot r_2 = \frac{c}{a}.$$

Step 2: Substitute the given roots $r_1 = -\frac{1}{2}$ and $r_2 = 3$:

$$\text{Sum of roots: } -\frac{1}{2} + 3 = \frac{5}{2}.$$

$$\text{Product of roots: } \left(-\frac{1}{2}\right)(3) = -\frac{3}{2}.$$

Step 3: Write the quadratic equation using the general form:

$$a \cdot x^2 - (\text{sum of roots}) \cdot x + (\text{product of roots}) = 0.$$

Substitute the values:

$$x^2 - \frac{5}{2}x - \frac{3}{2} = 0.$$

Step 4: Eliminate fractions by multiplying through by 2:

$$2x^2 - 5x - 3 = 0.$$

Therefore, the quadratic equation is $2x^2 - 5x - 3 = 0$.

2.4 Polynomials

Polynomials are algebraic expressions consisting of terms in the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are constants, x is the variable, and n is a non-negative integer.

Key Concepts:

- **Degree of a Polynomial:** The highest power of the variable in the polynomial. For example, the degree of $3x^4 + 2x^2 + 1$ is 4.
- **Addition and Subtraction:** Combine like terms to add or subtract polynomials.
- **Multiplication:** Use distributive property or expansion to multiply polynomials.
- **Factorization:** Rewrite a polynomial as a product of its factors using techniques like common factors, grouping, or special identities.

Example 1: Find the degree of the polynomial $4x^5 + 2x^3 - 7x + 8$.

Solution:

The highest power of x in the polynomial is 5.
Therefore, the degree of the polynomial is 5.

Example 2: Add the polynomials $2x^3 + 3x^2 - 5$ and $x^3 - 4x^2 + 7x + 1$.

Solution:

Align and add like terms:

$$(2x^3 + 3x^2 - 5) + (x^3 - 4x^2 + 7x + 1) = (2x^3 + x^3) + (3x^2 - 4x^2) + 7x + (-5 + 1).$$

Simplify:

$$3x^3 - x^2 + 7x - 4.$$

Therefore, the sum is $3x^3 - x^2 + 7x - 4$.

Example 3: Multiply the polynomials $(x + 2)$ and $(x^2 - 3x + 4)$.

Solution:

Use the distributive property:

$$(x + 2)(x^2 - 3x + 4) = x(x^2 - 3x + 4) + 2(x^2 - 3x + 4).$$

Expand:

$$x^3 - 3x^2 + 4x + 2x^2 - 6x + 8.$$

Combine like terms:

$$x^3 - x^2 - 2x + 8.$$

Therefore, the product is $x^3 - x^2 - 2x + 8$.

Example 4: Factorize the polynomial $x^3 + 3x^2 - x - 3$.

Solution:

Group terms and factorize:

$$x^3 + 3x^2 - x - 3 = (x^3 + 3x^2) - (x + 3).$$

Factor out common terms:

$$x^2(x + 3) - 1(x + 3).$$

Factorize further:

$$(x + 3)(x^2 - 1).$$

Use the difference of squares:

$$(x + 3)(x - 1)(x + 1).$$

Therefore, the factorized form is $(x + 3)(x - 1)(x + 1)$.

2.5 Remainder Arithmetic

Remainder arithmetic, also known as modular arithmetic, deals with integers under division by a fixed number (the modulus). It is commonly used in number theory and cryptography.

Key Concepts:

- **Modulo Operation:** The modulo operation finds the remainder when one integer is divided by another. For example, $11 \equiv 2 \pmod{3}$, because 11 gives the remainder 2 when 8 is divided by 3 ($\frac{11}{3} = 3 + \frac{2}{3}$).
  Formally, $a \equiv r \pmod{m}$ if $a = qm + r$, where q is the quotient and $0 \le r < m$.

- **Congruence Modulo:** Two integers a and b are congruent modulo m, written as $a \equiv b \pmod{m}$, if their difference is divisible by m. Formally:

$$a \equiv b \pmod{m} \iff m \mid (a - b).$$

- **Multiplication Table Modulo m:** A table showing the products of integers under modulo m, useful for understanding modular arithmetic operations.

Example 1: Verify if $17 \equiv 5 \pmod{6}$.

Solution:

Check if $17 - 5$ is divisible by 6:

$$17 - 5 = 12.$$

Since 12 is divisible by 6, we conclude:

$$17 \equiv 5 \pmod{6}.$$

Example 2: Compute $(7 + 10) \pmod{5}$.

Solution:

Add the numbers:

$$7 + 10 = 17.$$

Find the remainder when 17 is divided by 5:

$$17 \div 5 = 3 \text{ remainder } 2.$$

Therefore:

$$(7 + 10) \pmod{5} = 2.$$

Example 3: Compute $(9 \cdot 8) \pmod{7}$.

Solution:

Multiply the numbers:

$$9 \cdot 8 = 72.$$

Find the remainder when 72 is divided by 7:

$$72 \div 7 = 10 \text{ remainder } 2.$$

Therefore:

$$(9 \cdot 8) \pmod 7 = 2.$$

Example 4: Compute $3^4 \pmod 5$.

Solution:

Calculate the power:

$$3^4 = 81.$$

Find the remainder when 81 is divided by 5:

$$81 \div 5 = 16 \text{ remainder } 1.$$

Therefore:

$$3^4 \pmod 5 = 1.$$

Example 5: Multiplication Table Modulo 4.

Solution:

Construct the table for integers $0, 1, 2, 3$ modulo 4:

$\cdot \pmod 4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

This table shows the results of multiplication under modulo 4.

2.6 Sequences and Series

A sequence is an ordered list of numbers that follow a specific pattern, and a series is the sum of the terms of a sequence.

Key Concepts:

- **Sequence Definitions:**
  - **Sequence:** An ordered list of numbers arranged according to a specific rule or pattern. Each number in the sequence is called a term, and the position of a term is denoted by n. Sequences can be defined by a general formula (explicit rule) or a recursive formula. For example: 5, 10, 15, 20,... is a sequence with first term $a_1 = 5$, second term $a_2 = 10$ and so on.
  - **General Formula:** A formula that defines the nth term of a sequence, a_n, in terms of n. For examplme $a_n = n^2 - 2$ will give a sequence -1, 2, 7, 14, 23,...
  - **Recursive Formula:** A formula that defines each term of a sequence in relation to one or more previous terms. For example, $a_1 = 2$, $a_{n+1} = a_n + 3$ will give a sequence 2, 5, 8, 11,...

- **Arithmetic Progression (AP):** A sequence where each term increases or decreases by a constant value, called the common difference d. The general form is:

$$a, a + d, a + 2d, \ldots$$

  - **Nth Term of AP:**

$$a_n = a + (n - 1)d$$

    where a is the first term, d is the common difference, and n is the term number.
  - **Sum of the First n Terms of AP:**

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

- **Geometric Progression (GP):** A sequence where each term is multiplied by a constant value, called the common ratio r. The general form is:

$$a, ar, ar^2, \ldots$$

  - **Nth Term of GP:**

$$a_n = ar^{n-1}$$

    where a is the first term, r is the common ratio, and n is the term number.
  - **Sum of the First n Terms of GP:**

$$S_n = a\frac{1 - r^n}{1 - r}, \quad r \neq 1$$

  - **Sum to Infinity of GP:**

$$S_\infty = \frac{a}{1 - r}, \quad |r| < 1$$

- **Sum of Sequences:** The sum of the terms in a sequence, calculated using the formulas above depending on whether the sequence is arithmetic or geometric.

Example 1: Find the 10th term of the AP $3, 7, 11, \ldots$.

Solution:

Identify the values:

$$a = 3, \quad d = 7 - 3 = 4, \quad n = 10.$$

Use the formula for the nth term:

$$a_n = a + (n-1)d = 3 + (10-1) \cdot 4 = 3 + 36 = 39.$$

Therefore, the 10th term is 39.

Example 2: Find the sum of the first 8 terms of the AP $5, 9, 13, \ldots$.

Solution:

Identify the values:

$$a = 5, \quad d = 9 - 5 = 4, \quad n = 8.$$

Use the formula for the sum of the first n terms:

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{8}{2}[2 \cdot 5 + (8-1) \cdot 4].$$

Simplify:

$$S_n = 4[10 + 28] = 4 \cdot 38 = 152.$$

Therefore, the sum of the first 8 terms is 152.

Example 3: Find the 6th term of the GP $2, 4, 8, \ldots$.

Solution:

Identify the values:

$$a = 2, \quad r = 4 \div 2 = 2, \quad n = 6.$$

Use the formula for the nth term:

$$a_n = ar^{n-1} = 2 \cdot 2^{6-1} = 2 \cdot 2^5 = 2 \cdot 32 = 64.$$

Therefore, the 6th term is 64.

Example 4: Find the sum to infinity of the GP $3, 1.5, 0.75, \ldots$.

Solution:

Identify the values:

$$a = 3, \quad r = 1.5 \div 3 = 0.5.$$

Use the formula for the sum to infinity:

$$S_\infty = \frac{a}{1-r} = \frac{3}{1-0.5} = \frac{3}{0.5} = 6.$$

Therefore, the sum to infinity is 6.

2.7 Matrices and Determinants

Matrices are rectangular arrays of numbers arranged in rows and columns, used to represent and solve systems of linear equations and perform transformations in geometry.

Key Concepts:

- **Matrix Definition:** A matrix is an array of numbers, symbols, or expressions arranged in rows and columns. For example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- **Size of a Matrix:** The size of a matrix is defined as the number of rows $\times$ the number of columns. For example, the above matrix A is a 3×3 matrix.

- **Basic Operations with Matrices:**
  - **Addition and Subtraction:** Matrices of the same size are added or subtracted element by element.
  - **Multiplication:** To multiply two matrices, the number of columns in the first matrix must equal the number of rows in the second matrix.
    The element in the ith row and jth column of the product matrix is the sum of the products of the corresponding elements of the ith row of the first matrix and the jth column of the second matrix.

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

- **Determinant:** The determinant is a scalar value associated with a square matrix, used to determine whether the matrix is invertible. For a 2×2 matrix:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{then } \det(A) = ad - bc.$$

- **Inverse Matrix:** The inverse of a square matrix A, denoted A^{-1}, exists if and only if $\det(A) \neq 0$. For a 2×2 matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 1: Add the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

Solution:

Add corresponding elements:

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

Therefore, $A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$.

Example 2: Find the determinant of $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$.

Solution:

Use the formula for the determinant of a 2×2 matrix:

$$\det(A) = (3)(1) - (4)(2) = 3 - 8 = -5.$$

Therefore, $\det(A) = -5$.

Example 3: Multiply the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

Solution:

Use the rule for matrix multiplication:

$$A \cdot B = \begin{bmatrix} (1)(2) + (2)(1) & (1)(0) + (2)(3) \\ (3)(2) + (4)(1) & (3)(0) + (4)(3) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}.$$

Therefore, $A \cdot B = \begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}$.

Example 4: Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

Solution:

Step 1: Calculate the determinant:

$$\det(A) = (2)(4) - (3)(1) = 8 - 3 = 5.$$

Step 2: Use the formula for the inverse of a 2×2 matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}.$$

Therefore:

$$A^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

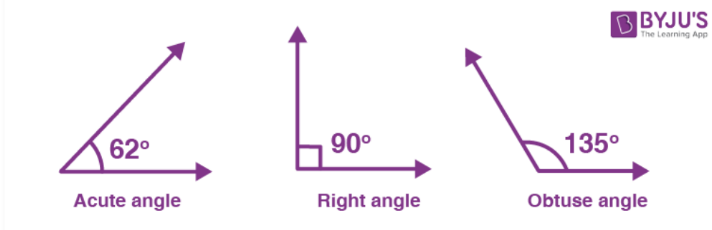
Chapter 3: Geometry and Mensuration

3.1 Plane Geometry

Plane geometry deals with shapes, lines, and angles in two-dimensional space. Understanding the properties of various geometric figures is fundamental to solving problems in geometry.

Key Concepts:

Triangle and its Properties: A triangle is a three-sided polygon with three angles. The sum of the interior angles of a triangle is always 180°. Triangles can be classified based on their sides (scalene, isosceles, equilateral) or angles (acute, right, obtuse). The exterior angle of a triangle is equal to the sum of the two opposite interior angles.



Pythagoras Theorem: In a right-angled triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. Mathematically:

$$c^2 = a^2 + b^2,$$

where c is the hypotenuse, and a and b are the other two sides.

Quadrilaterals and Their Properties: A quadrilateral is a four-sided polygon with four angles. Common types include squares, rectangles, parallelograms, trapeziums, and rhombuses. The sum of the interior angles of a quadrilateral is 360°. Specific properties include:

- Opposite sides of a rectangle and parallelogram are equal.
- Diagonals of a square bisect each other at right angles.
- A trapezium has one pair of parallel sides.

Inscribed Triangles and Polygons: An inscribed polygon is one whose vertices all lie on a single circle, called the circumcircle. The center of this circle is called the circumcenter, and the radius is the circumradius. For inscribed triangles, the angle subtended by a diameter at the circumference is always a right angle.

Internal and External Angles of Triangles and Polygons: The sum of the internal angles of a polygon with n sides is:

$$(n-2) \times 180°.$$

The sum of the external angles of any polygon is always 360°, making each exterior angle equal to

$$\frac{360°}{n}.$$

Polygons and Their Properties: Polygons are closed figures with three or more sides. Regular polygons have equal sides and angles. Examples include pentagons, hexagons, and octagons. The measure of each internal angle of a regular polygon with n sides is:

$$\frac{(n-2)\times 180^\circ}{n}.$$

Examples:

Example 1: Calculate the hypotenuse of a right triangle with sides 3 cm and 4 cm.

Solution:

Use Pythagoras theorem:

$$c^2 = a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25.$$

$$c = \sqrt{25} = 5 \text{ cm}.$$

Therefore, the hypotenuse is 5 cm.

Example 2: Find the sum of the internal angles of a hexagon.

Solution:

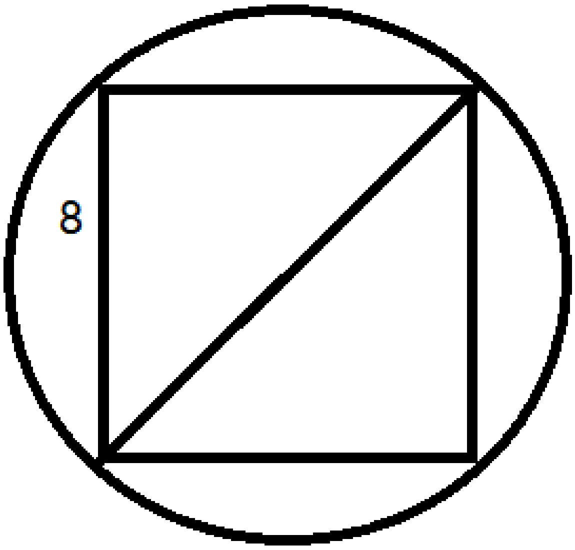
For hexagon n=6.
Use the formula for the sum of internal angles of a polygon:

$$(n-2)\times 180^\circ = (6-2)\times 180^\circ = 4\times 180^\circ = 720^\circ.$$

Therefore, the sum of the internal angles of a hexagon is 720°.

Example 3: A square is inscribed in a circle. Find the radius of the circle if the side of the square is 8 cm.

Solution:



The diagonal of the square is the diameter of the circle. Use the Pythagoras theorem:

$$diagonal^2 = side^2 + side^2 = 8^2 + 8^2 = 64 + 64 = 128.$$

$$diagonal = \sqrt{128} = 11.31 \text{ cm.}$$

The radius is half the diagonal:

$$radius = \frac{11.31}{2} = 5.66 \text{ cm.}$$

Therefore, the radius of the circle is 5.66 cm.

Example 4: Find each internal angle of a regular pentagon.

Solution:

Use the formula for each internal angle of a regular polygon with n=5:

$$\frac{(n-2)\times 180^\circ}{n} = \frac{(5-2)\times 180^\circ}{5} = \frac{3\times 180^\circ}{5} = \frac{540^\circ}{5} = 108^\circ.$$

Therefore, each internal angle of a regular pentagon is 108°.

3.2 Circles

Circles are fundamental shapes in geometry and have unique properties and theorems associated with them. Below are key concepts and theorems related to circles.

Key Concepts:

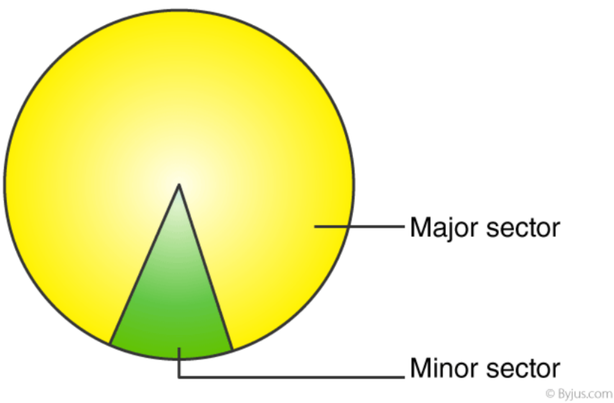
- **Radius and Diameter:** The radius is the distance from the center of the circle to any point on its circumference. The diameter is twice the radius.
- **Circumference:** The perimeter of a circle is given by the formula:

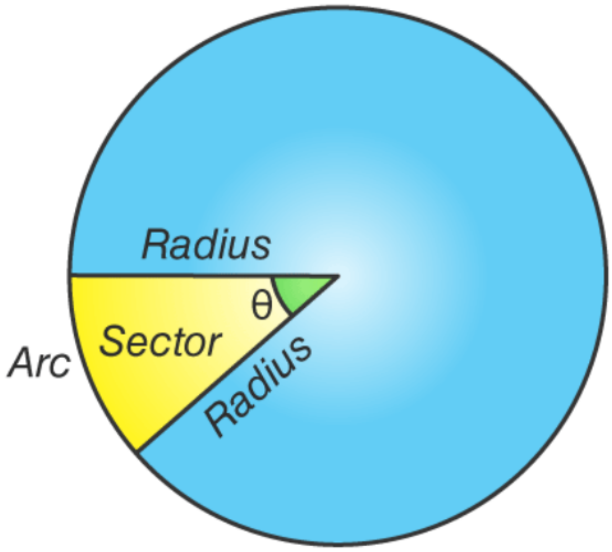
$$C = 2\pi r,$$

  where r is the radius.
- **Area:** The area of a circle is given by:

$$A = \pi r^2.$$

- **Tangent:** A tangent to a circle is a line that touches the circle at exactly one point. The radius at the point of tangency is perpendicular to the tangent.
- **Chord:** A chord is a line segment with endpoints on the circle. The perpendicular from the center to a chord bisects the chord.
- **Arc and Sector:** An arc is a part of the circumference, and a sector is the region enclosed by two radii and an arc.





- **Area of an sector and Arc length:**

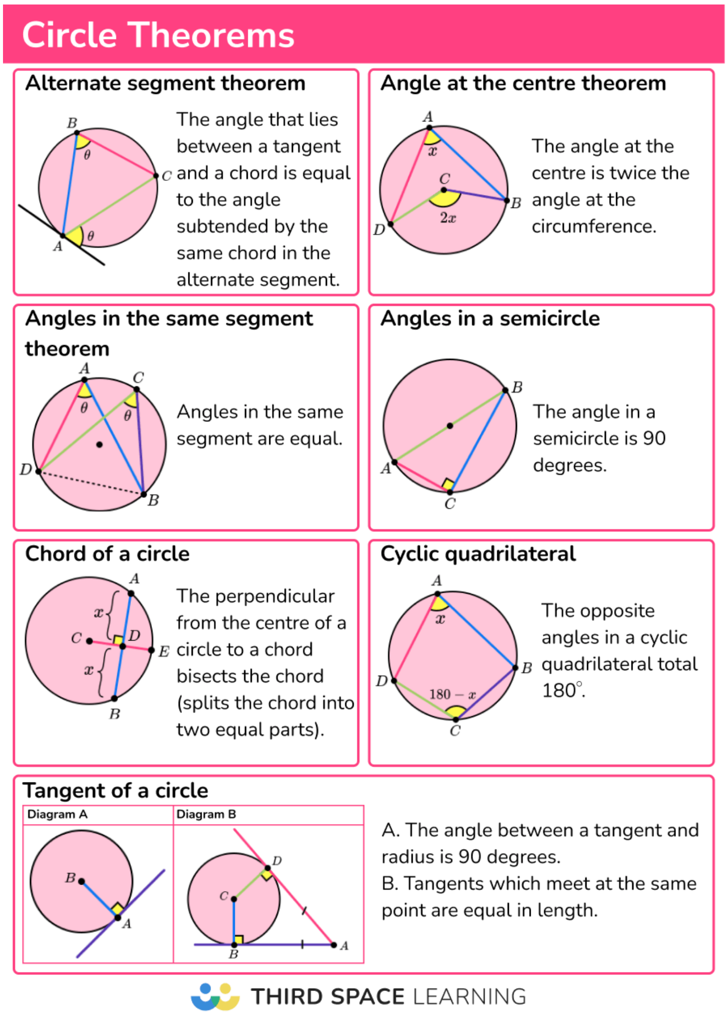
$$\text{Area of Sector} = \frac{\theta}{360^\circ} \times \pi r^2,$$

$$\text{Arc Length} = \frac{\theta}{360^\circ} \times 2\pi r,$$

where θ is the central angle in degrees and r is the radius of the circle.

Circle Theorems:

- **Angle at the Center:** The angle subtended by an arc at the center of a circle is twice the angle subtended at any point on the circumference.
- **Angle in a Semicircle:** The angle subtended by a diameter at the circumference is a right angle.
- **Cyclic Quadrilateral:** The opposite angles of a cyclic quadrilateral (a quadrilateral inscribed in a circle) are supplementary.
- **Tangent-Secant Theorem:** If a tangent and a secant are drawn from an external point, then the square of the length of the tangent is equal to the product of the secant's external and total lengths.



Examples:

Example 1: Find the circumference of a circle with radius 7 cm.

Solution:

Use the formula for circumference:

$$C = 2\pi r = 2 \times \pi \times 7 = 14\pi \approx 43.98 \text{ cm}.$$

Therefore, the circumference is approximately 43.98 cm.

Example 2: A chord of a circle is 12 cm long, and its distance from the center is 5 cm. Find the radius of the circle.

Solution:

Let the radius be r. Using the Pythagoras theorem in the triangle formed by the radius, half the chord, and the perpendicular distance:

$$r^2 = 6^2 + 5^2 = 36 + 25 = 61.$$

$$r = \sqrt{61} \approx 7.81 \text{ cm}.$$

Therefore, the radius of the circle is approximately 7.81 cm.

Example 3: Prove that the angle subtended by a diameter at the circumference is a right angle.

Solution:

Let ABC be a triangle inscribed in a circle, where AB is the diameter. By the angle at the center theorem:

$$\text{Angle at the center } \angle AOB = 180^\circ.$$

$$\text{Angle at the circumference } \angle ACB = \frac{1}{2} \times 180^\circ = 90^\circ.$$

Therefore, the angle subtended by a diameter is always a right angle.

Example 4: Find the area of a sector with radius 6 cm and central angle 60°.

Solution:

Use the formula for the area of a sector:

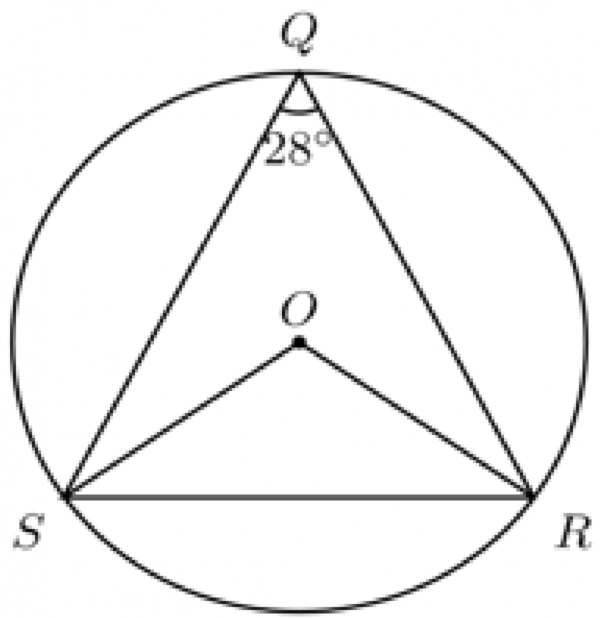
$$A = \frac{\theta}{360^\circ} \times \pi r^2 = \frac{60^\circ}{360^\circ} \times \pi \times 6^2.$$

Simplify:

$$A = \frac{1}{6} \times \pi \times 36 = 6\pi \approx 18.85 \text{ cm}^2.$$

Therefore, the area of the sector is approximately 18.85 cm^2.

Example 5: In the diagram below, O is the center of the circle QRS and $\angle SQR = 28^\circ$. Find $\angle ORS$.



Solution:

Step 1: Recall the Circle Theorem for angles at the center and circumference: The angle subtended by an arc at the center of a circle is twice the angle subtended at the circumference by the same arc.
Step 2: Identify the arc and the angles: - $\angle SQR$ is the angle subtended at the circumference by the arc SR. - $\angle SOR$ is the angle subtended at the center by the same arc.
Step 3: Use the Circle Theorem:

$$\angle SOR = 2 \times \angle SQR.$$

Substitute $\angle SQR = 28^\circ$:

$$\angle SOR = 2 \times 28^\circ = 56^\circ.$$

Step 4: Find $\angle ORS$: Since $\triangle OSR$ is an isosceles triangle (radii OS and OR are equal), the base angles are equal:

$$\angle ORS = \angle OSR = \frac{180^\circ - \angle SOR}{2}.$$

Substitute $\angle SOR = 56^\circ$:

$$\angle ORS = \frac{180^\circ - 56^\circ}{2} = \frac{124^\circ}{2} = 62^\circ.$$

Therefore, $\angle ORS = 62^\circ$.

3.3 Longitude and Latitude Problems, Bearings

Longitude and latitude are used to specify the position of points on the Earth's surface. Bearings and directions play a crucial role in navigation and solving geographical problems.

Key Concepts:

- **Bearing of an Angle:** Bearings are used to describe the direction of one point relative to another. Bearings are measured clockwise from the north direction and are usually expressed in three digits. For example, 045° represents a bearing of 45° from north.

- **Main Directions:** The primary compass directions are:
  - **North (N):** 0° or 360°
  - **East (E):** 90°
  - **South (S):** 180°
  - **West (W):** 270°

Examples:

Example 1: A ship travels on a bearing of 120°. What direction is the ship moving?

Solution:

A bearing of 120° is measured clockwise from north. This places the direction in the southeast quadrant. Therefore, the ship is moving southeast.

Example 2: Find the bearing from Point A to Point B if A is due west of B.

Solution:

If A is due west of B, the bearing from A to B is measured clockwise from north:

$$\text{Bearing} = 270^\circ.$$

Therefore, the bearing is 270°.

Example 3: John was facing $S35°E$. If he turned $90°$ in the anticlockwise direction, find his new direction.

Solution:

Step 1: Convert $S35°E$ to a bearing:

$$\text{Bearing} = 180° - 35° = 145°.$$

Step 2: Subtract $90°$ for the anticlockwise turn:

$$145° - 90° = 55°.$$

Step 3: Convert the bearing back to a compass direction:

$$55° \text{ is measured as } N55°E.$$

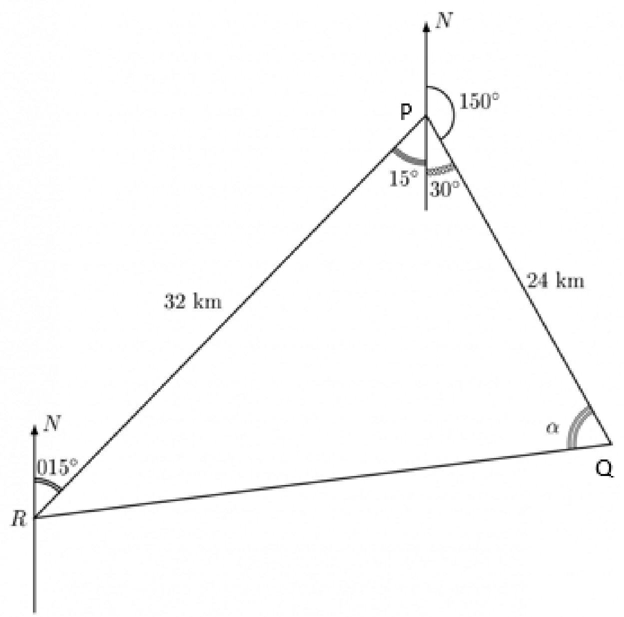
Therefore, John's new direction is $N55°E$ (bearing of $55°$).

Example 3: The bearing of Q from P is 015° and the bearing of P from R is 015°. If Q and R are 24 km and 32 km respectively from P:

(i) Represent this information in a diagram.

(ii) Calculate the distance between Q and R, correct to two decimal places.

(iii) Find the bearing of R from Q, correct to the nearest degree.



Solution:

Step 1: Represent the given bearings in a diagram. Since both bearings are measured from the north, we construct a diagram accordingly.
Step 2: Use the cosine rule to find the distance between Q and R. The angle between PQ and PR is:

$$\theta = (15^\circ + 35^\circ) = 145^\circ.$$

Using the cosine rule:

$$|QR|^2 = 32^2 + 24^2 - 2 \times 32 \times 24 \times \cos 45^\circ$$

$$|QR|^2 = 1024 + 576 - 1536 \cos 45^\circ$$

$$= 1600 - 1086.1056$$

$$|QR|^2 = 513.8944$$

$$|QR| = \sqrt{513.8944} = 22.669 \text{ km}$$

$$\approx 22.67 \text{ km to 2 dp}$$

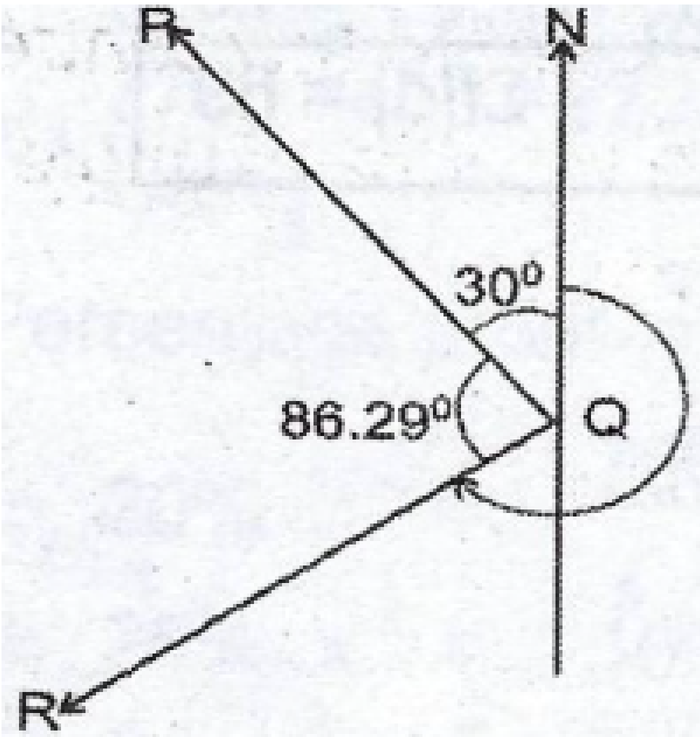
Step 3: Find the angle $\angle PQRusingthesinerule : \frac{32}{\sin \angle PQR} = \frac{22.67}{\sin 45^\circ}$

$$\sin \angle PQR = \frac{32 \times \sin 45^\circ}{22.67}$$

$$= 0.9981$$

$$\angle PQR = \sin^{-1}(0.9981) = 86.4787^\circ$$

Step 4: Calculate the bearing of R from Q:



The bearing of R from Q is given by the reflex angle NQR. Thus,

$$\text{Reflex } \angle NQR = 360^\circ - (86.47^\circ + 30^\circ)$$

$$= 360^\circ - 116.47^\circ$$

$$= 243.53^\circ$$

Hence, the bearing of R from Q:

$$\approx 244^\circ \quad \text{(to the nearest degree)}$$

3.4 Coordinate Geometry

Coordinate geometry involves the study of geometric figures using the coordinate plane and algebraic techniques. It is fundamental for solving problems involving points, lines, and circles.

Key Concepts:

- **Slope (Gradient) of a Line:** The slope of a line measures its steepness and is calculated as:
$$m = \frac{y_2 - y_1}{x_2 - x_1},$$
where (x_1, y_1) and (x_2, y_2) are two points on the line.

- **Equation of a Line:**
  - **Slope-Intercept Form:** $y = mx + c$, where m is the slope and c is the y-intercept.
  - **Point-Slope Form:** $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line.
  - **Two-Intercept Form:** $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are the x- and y-intercepts, respectively.

- **Parallel and Perpendicular Lines:**
  - Parallel lines have the same slope: $m_1 = m_2$.
  - Perpendicular lines have slopes that multiply to -1: $m_1 \cdot m_2 = -1$.

- **Midpoint of a Line Segment:** The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is:
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

- **Distance Between Two Points:** The distance between two points (x_1, y_1) and (x_2, y_2) is:
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- **Equation of a Circle:** The equation of a circle with center (h, k) and radius r is:
$$(x - h)^2 + (y - k)^2 = r^2.$$

- **Finding the Equation of a Line Parallel to a Given Line:** A line parallel to $y = mx + c$ has the same slope m but a different y-intercept.

- **Finding the Equation of a Line Perpendicular to a Given Line:** A line perpendicular to $y = mx + c$ has a slope of $-\frac{1}{m}$.

Examples:

Example 1: Find the slope of the line passing through the points $(2, 3)$ and $(5, 7)$.

Solution:

Use the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 3}{5 - 2} = \frac{4}{3}.$$

Therefore, the slope is $\frac{4}{3}$.

Example 2: Write the equation of the line with slope 2 passing through the point $(3, 4)$.

Solution:

Use the point-slope form:

$$y - y_1 = m(x - x_1).$$

Substitute $m = 2$, $x_1 = 3$, and $y_1 = 4$:

$$y - 4 = 2(x - 3).$$

Simplify:

$$y = 2x - 6 + 4 = 2x - 2.$$

Therefore, the equation is $y = 2x - 2$.

Example 3: Find the distance between the points $(1, 2)$ and $(4, 6)$.

Solution:

Use the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Substitute the values:

$$d = \sqrt{(4 - 1)^2 + (6 - 2)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Therefore, the distance is 5 units.

Example 4: Find the equation of a circle with center $(2, -3)$ and radius 5.

Solution:

Use the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2.$$

Substitute $h = 2$, $k = -3$, and $r = 5$:

$$(x - 2)^2 + (y + 3)^2 = 25.$$

Therefore, the equation of the circle is:

$$(x - 2)^2 + (y + 3)^2 = 25.$$

Example 5: Find the equation of a line perpendicular to $y = \frac{1}{2}x + 3$ that passes through the point $(4, 2)$.

Solution:

Step 1: Find the slope of the perpendicular line:

$$m_{\text{perpendicular}} = -\frac{1}{m_{\text{original}}} = -\frac{1}{\frac{1}{2}} = -2.$$

Step 2: Use the point-slope form:

$$y - y_1 = m(x - x_1).$$

Substitute $m = -2$, $x_1 = 4$, and $y_1 = 2$:

$$y - 2 = -2(x - 4).$$

Simplify:

$$y = -2x + 8 + 2 = -2x + 10.$$

Therefore, the equation of the line is $y = -2x + 10$.

Example 6: The x- and y-intercepts of a straight line are $-\frac{3}{4}$ and $\frac{2}{7}$, respectively. Find the equation of the line.

Solution:

Step 1: Use the two-intercept form of a line:

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where a is the x-intercept and b is the y-intercept.
Step 2: Substitute $a = -\frac{3}{4}$ and $b = \frac{2}{7}$:

$$\frac{x}{-\frac{3}{4}} + \frac{y}{\frac{2}{7}} = 1.$$

Step 3: Simplify the fractions:

$$-\frac{4x}{3} + \frac{7y}{2} = 1.$$

Step 4: Eliminate the fractions by multiplying through by the least common denominator (LCD), which is 6:

$$6 \times \left(-\frac{4x}{3}\right) + 6 \times \left(\frac{7y}{2}\right) = 6 \cdot 1.$$

Simplify:

$$-8x + 21y = 6.$$

Therefore, the equation of the line is:

$$-8x + 21y = 6.$$

Alternatively, in point-slope form: Using the slope $m = \frac{\Delta y}{\Delta x} = \frac{\frac{2}{7}-0}{0-\left(-\frac{3}{4}\right)} = \frac{\frac{2}{7}}{\frac{3}{4}} = \frac{8}{21}$, and passing through the point $(0, \frac{2}{7})$, the equation becomes:

$$y - \frac{2}{7} = \frac{8}{21}(x - 0).$$

Therefore, the gradient-point form is:

$$y = \frac{8}{21}x + \frac{2}{7}.$$

3.5 Mensuration

Mensuration involves calculating lengths, areas, surface areas, and volumes of various geometric shapes. It is divided into two categories: 2D shapes (plane figures) and 3D shapes (solids).

Key Concepts:

1. Areas and Perimeters of 2D Shapes:

- **Triangle:**

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}, \quad \text{Perimeter} = a + b + c,$$

where a, b, c are the lengths of the sides.

- **Square:**

$$\text{Area} = \text{side}^2, \quad \text{Perimeter} = 4 \times \text{side}.$$

- **Rectangle:**

$$\text{Area} = \text{length} \times \text{breadth}, \quad \text{Perimeter} = 2 \times (\text{length} + \text{breadth}).$$

- **Trapezium:**

$$\text{Area} = \frac{1}{2} \times (\text{sum of parallel sides}) \times \text{height}.$$

- **Parallelogram:**

$$\text{Area} = \text{base} \times \text{height}, \quad \text{Perimeter} = 2 \times (\text{length} + \text{breadth}).$$

- **Circle:**

$$\text{Area} = \pi r^2, \quad \text{Circumference} = 2\pi r,$$

where r is the radius.

- **Circular Segment:**

$$\text{Area of Segment} = \frac{r^2}{2}(\theta - \sin\theta), \quad \text{Arc Length} = r\theta,$$

where r is the radius and θ is the angle in radians.

2. Surface Areas and Volumes of 3D Shapes:

- **Cube:**

$$\text{Surface Area} = 6a^2, \quad \text{Volume} = a^3,$$

where a is the side length.

- **Cuboid:**

$$\text{Surface Area} = 2(lb + bh + hl), \quad \text{Volume} = l \times b \times h,$$

where l, b, h are the length, breadth, and height.

- **Cylinder:**

  $$\text{Curved Surface Area} = 2\pi rh, \quad \text{Total Surface Area} = 2\pi r(h+r), \quad \text{Volume} = \pi r^2 h,$$

  where r is the radius and h is the height.

- **Cone:**

  $$\text{Curved Surface Area} = \pi rl, \quad \text{Total Surface Area} = \pi r(l+r), \quad \text{Volume} = \frac{1}{3}\pi r^2 h,$$

  where r is the radius, h is the height, and l is the slant height.

- **Sphere:**

  $$\text{Surface Area} = 4\pi r^2, \quad \text{Volume} = \frac{4}{3}\pi r^3,$$

  where r is the radius.

Relation Between Mass, Density, and Volume: The relationship is given by:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}, \quad \text{Mass} = \text{Density} \times \text{Volume}.$$

Examples:

Example 1: Find the area and perimeter of a rectangle with length 8 cm and breadth 5 cm.

Solution:

Area:

$$\text{Area} = \text{length} \times \text{breadth} = 8 \times 5 = 40 \text{ cm}^2.$$

Perimeter:

$$\text{Perimeter} = 2 \times (\text{length} + \text{breadth}) = 2 \times (8 + 5) = 26 \text{ cm}.$$

Therefore, the area is 40 cm^2 and the perimeter is 26 cm.

Example 2: Calculate the volume and total surface area of a sphere with radius 7 cm.

Solution:

Volume:

$$\text{Volume} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(7)^3 = \frac{4}{3} \times \pi \times 343 \approx 1436.76 \text{ cm}^3.$$

Surface Area:

$$\text{Surface Area} = 4\pi r^2 = 4 \times \pi \times (7)^2 = 4 \times \pi \times 49 \approx 615.75 \text{ cm}^2.$$

Therefore, the volume is approximately 1436.76 cm^3 and the surface area is approximately 615.75 cm^2.

Example 3: Find the mass of a cylindrical object with radius 3 cm, height 10 cm, and density 8 g/cm^3.

Solution:

Volume:

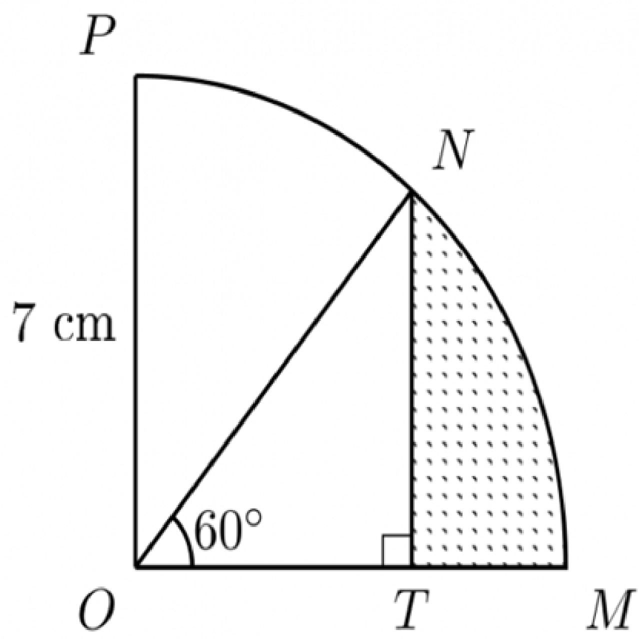
$$\text{Volume} = \pi r^2 h = \pi(3)^2(10) = 90\pi \approx 282.74 \text{ cm}^3.$$

Mass:

$$\text{Mass} = \text{Density} \times \text{Volume} = 8 \times 282.74 = 2261.92 \text{ g}.$$

Therefore, the mass is approximately 2261.92 g.

Example 4: In the diagram below, the radius of the sector of circle center O is $7\,$cm and $\angle MON = 60°$. Find, correct to one decimal place, the area of the shaded portion. (Take $\pi = \frac{22}{7}$).



Solution:

Step 1: Calculate the area of the sector:

$$\text{Area of the sector} = \frac{\theta}{360°} \times \pi r^2.$$

Substitute $\theta = 60°$, $r = 7$, and $\pi = \frac{22}{7}$:

$$\text{Area of the sector} = \frac{60}{360} \times \frac{22}{7} \times 7^2.$$

Simplify:

$$\text{Area of the sector} = \frac{1}{6} \times \frac{22}{7} \times 49 = \frac{22 \times 49}{42} = \frac{1078}{42} \approx 25.7\,\text{cm}^2.$$

Step 2: Calculate the area of $\triangle OMT$: Since $\triangle OMT$ is a right triangle:

$$\text{Area of } \triangle OMT = \frac{1}{2} \times \text{base} \times \text{height}.$$

Here, base $= 7\,\text{cm}$ and height $= 7\,\text{cm}$:

$$\text{Area of } \triangle OMT = \frac{1}{2} \times 7 \times 7 = 24.5\,\text{cm}^2.$$

Step 3: Find the area of the shaded portion:

$$\text{Area of shaded portion} = \text{Area of sector} - \text{Area of } \triangle OMT.$$

Substitute:

$$\text{Area of shaded portion} = 25.7 - 24.5 = 1.2\,\text{cm}^2.$$

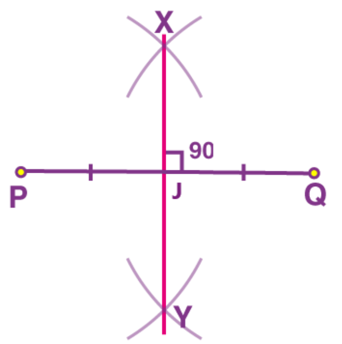
Therefore, the area of the shaded portion is approximately $1.2\,\text{cm}^2$.

3.6 Loci

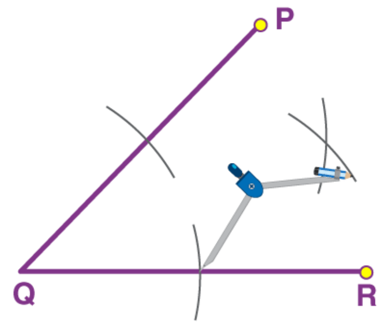
Loci are the set of points that satisfy a specific condition or rule. They are used to describe the path traced by moving points under given constraints in geometry.

Key Concepts:

- **Definition of a Locus:** A locus is the path traced by a point that moves according to a specific condition or rule.
- **Some Standard Loci:**
  - **Locus of a Point at a Fixed Distance from Another Point:** This is a circle with the fixed point as the center and the given distance as the radius.
  - **Locus of Points Equidistant from Two Points:** This is the perpendicular bisector of the line segment joining the two points. Perpendicular bisector to a line PQ is a line passing through midpoint of P and Q and perpendicular to PQ.



  - **Locus of Points Equidistant from Two Intersecting Lines:** This is the angle bisector of the angles formed by the intersecting lines.



  - **Locus of a Point at a Fixed Distance from a Line:** This consists of two parallel lines at the given distance from the original line, one on either side.
- **Real-Life Applications:** Loci are applied in navigation, construction, and design to determine paths or boundaries.

Examples:

Example 1: Describe the locus of points equidistant from two fixed points A and B.

Solution:

The locus of points equidistant from A and B is the perpendicular bisector of the line segment joining A and B.

Example 2: Describe the locus of points equidistant from two intersecting lines.

Solution:

The locus of points equidistant from two intersecting lines is the pair of angle bisectors of the angles formed by the lines.

Example 3: A point P moves such that it is always $5\,\text{cm}$ away from a fixed point O. Describe and sketch the locus of P.

Solution:

The locus of P is a circle with center O and radius $5\,\text{cm}$.

Example 4: A point P moves such that it is always $3\,\text{cm}$ away from a straight line L. Describe and sketch the locus of P.

Solution:

The locus of P consists of two parallel lines, each $3\,\text{cm}$ away from L, one on each side.

Example 5: A point P moves such that it is equidistant from two fixed points A and B and also equidistant from two intersecting lines L_1 and L_2. Describe the locus of P.

Solution:

The locus of P is the intersection of the perpendicular bisector of the line segment AB and the angle bisectors of the angles formed by the intersecting lines L_1 and L_2.

3.7 Construction

Construction in geometry involves creating geometric shapes, angles, and lines using a ruler and a compass. It is a fundamental skill for solving problems that require precision and accuracy.

Key Concepts:

- **Basic Tools for Construction:**
  - **Ruler:** Used to draw straight lines and measure distances.
  - **Compass:** Used to draw circles and arcs and to measure distances that can be transferred onto diagrams.
- **Constructing a Perpendicular Bisector:** The perpendicular bisector of a line segment divides it into two equal parts and forms a $90°$ angle with the line segment.
- **Bisecting an Angle:** The angle bisector divides an angle into two equal parts.
- **Constructing a Triangle:** A triangle can be constructed given the following:
  - Three sides (SSS).
  - Two sides and the included angle (SAS).
  - Two angles and one side (ASA).
- **Constructing Loci:** Using a compass and ruler, loci of points can be constructed based on specific geometric conditions, such as points equidistant from a fixed point or line.

Examples:

Example 1: Construct the perpendicular bisector of a line segment AB.

Solution:

1. Place the compass at A and draw an arc above and below the line segment.
2. Without changing the compass width, place the compass at B and draw arcs above and below the line segment, intersecting the first arcs.
3. Use the ruler to draw a straight line through the points of intersection.
The resulting line is the perpendicular bisector of AB.

Example 2: Bisect a $60°$ angle.

Solution:

1. Draw a $60°$ angle using a protractor. Label the vertex O.
2. Place the compass at O and draw an arc intersecting both arms of the angle. Label the points of intersection A and B.
3. Place the compass at A and B and draw two arcs that intersect each other inside the angle. Label the point of intersection P.
4. Draw a straight line from O to P.
The line OP bisects the $60°$ angle into two $30°$ angles.

Example 3: Construct a triangle given sides $AB = 5\,\text{cm}$, $AC = 4\,\text{cm}$, and $BC = 6\,\text{cm}$.

Solution:

1. Draw a base line $BC = 6\,\text{cm}$.
2. Place the compass at B and draw an arc of radius $5\,\text{cm}$.
3. Place the compass at C and draw an arc of radius $4\,\text{cm}$.
4. Label the point of intersection of the arcs as A.
5. Join A to B and A to C with straight lines.

The resulting triangle ABC satisfies the given dimensions.

Example 4: Construct the locus of points $3\,\text{cm}$ away from a given line L.

Solution:

1. Place the compass at a point on L and draw arcs of radius $3\,\text{cm}$ on both sides of the line.
2. Repeat the process at multiple points along L.
3. Use the ruler to draw two parallel lines through the arc intersections.

The resulting lines are the locus of points $3\,\text{cm}$ away from L.

3.8 Transformation Geometry

Transformation geometry involves moving, rotating, reflecting, enlarging, or shearing objects while preserving certain properties.

Key Concepts:

- **Translation:** Moving a shape without rotating or resizing it. A translation is represented as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+a \\ y+b \end{pmatrix}$$

  where (a, b) is the translation vector.

- **Rotation:** Turning a shape around a fixed point (center of rotation). A rotation by 90° counterclockwise about the origin transforms a point (x, y) to:

$$(x', y') = (-y, x).$$

- **Reflection:** Flipping a shape over a line (mirror line). Common reflections include:

$$\text{Reflection over } y = 0 \Rightarrow (x, y) \to (x, -y)$$

$$\text{Reflection over } x = 0 \Rightarrow (x, y) \to (-x, y)$$

- **Enlargement:** Resizing a shape by a scale factor k about a center (a, b). The transformation is:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a + k(x-a) \\ b + k(y-b) \end{pmatrix}$$

- **Shear:** A transformation that distorts a shape along a particular axis while keeping one line fixed. Horizontal shear is given by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + ky \\ y \end{pmatrix}$$

  where k is the shear factor.

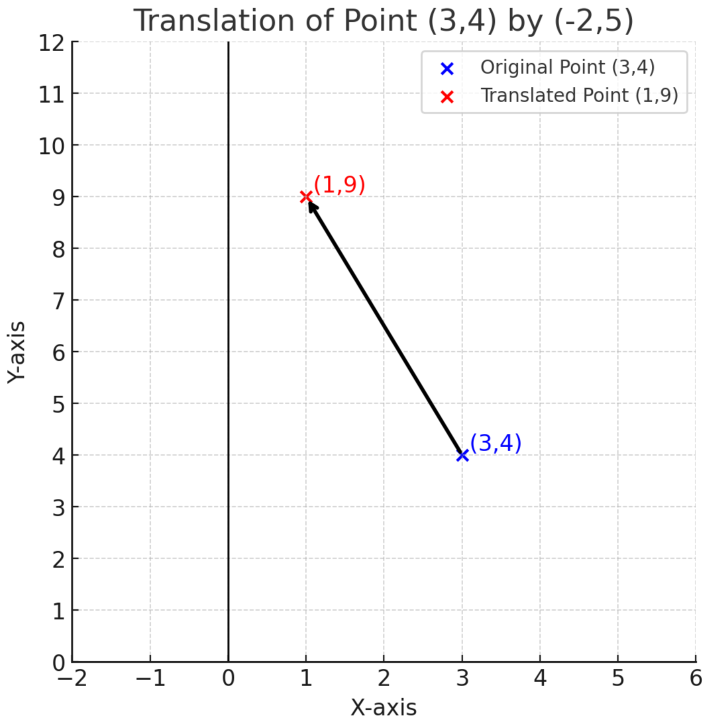
Examples:

Example 1: Find the image of the point $(3, 4)$ under the translation $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

Solution:

Step 1: Apply the translation rule.

$$(x', y') = (3 - 2, 4 + 5) = (1, 9).$$



Thus, the new coordinates are **$(1, 9)$**.

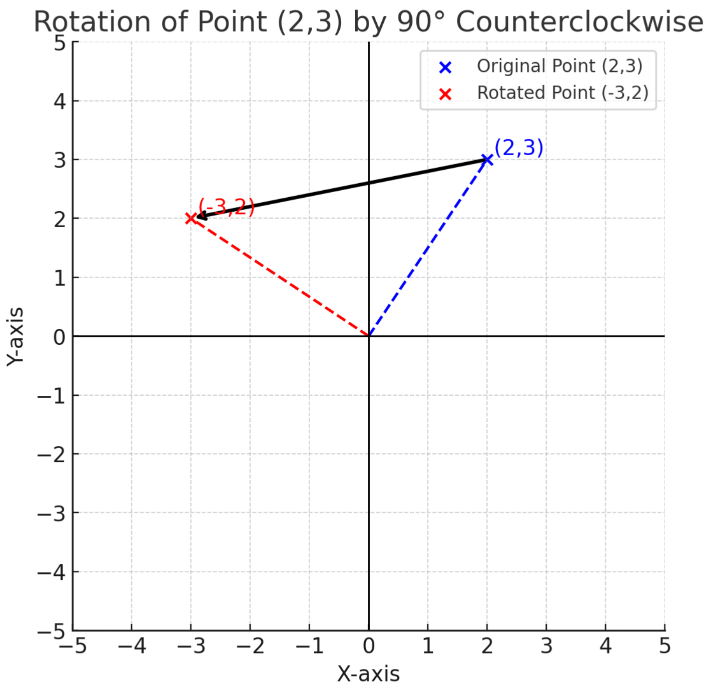
Example 2: Rotate the point $(2, 3)$ by $90°$ counterclockwise about the origin.

Solution:

Using the $90°$ rotation formula:

$$(x', y') = (-y, x).$$

$$(x', y') = (-3, 2).$$

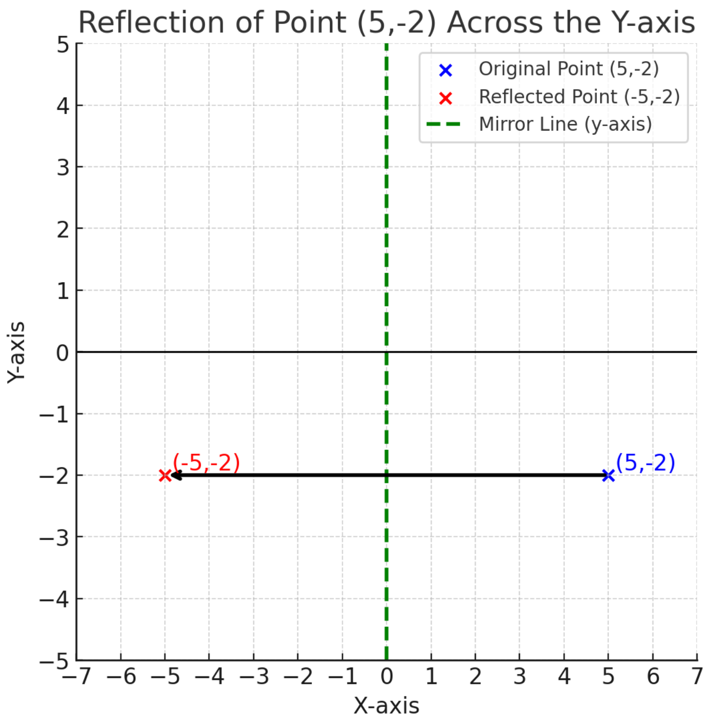


Thus, the new coordinates are **$(-3, 2)$**.

Example 3: Reflect the point $(5, -2)$ across the y-axis.
Solution:
Reflection over the y-axis:

$$(x', y') = (-x, y).$$

$$(x', y') = (-5, -2).$$

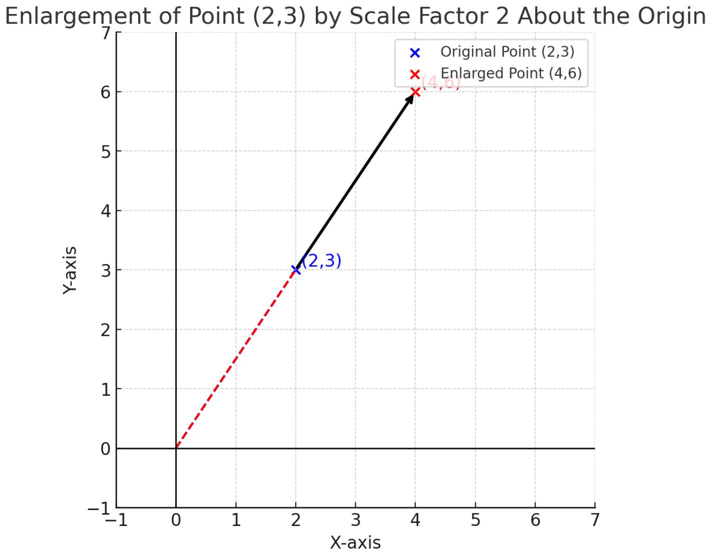


Thus, the new coordinates are **$(-5, -2)$**.

Example 4: Enlarge the point $(2, 3)$ by scale factor $k = 2$ about the origin.
Solution:
Using the enlargement formula:

$$(x', y') = (kx, ky).$$

$$(x', y') = (2 \times 2, 2 \times 3) = (4, 6).$$

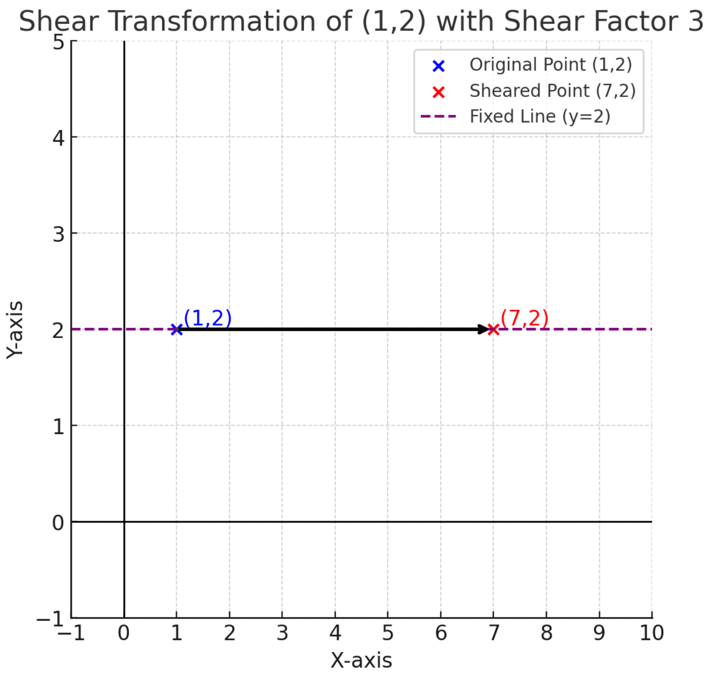


Thus, the new coordinates are **$(4, 6)$**.

Example 5: Apply a shear transformation with shear factor $k = 3$ along the x-axis to the point $(1, 2)$.
Solution:
Using the shear formula:

$$(x', y') = (x + ky, y).$$

$$(x', y') = (1 + 3(2), 2) = (7, 2).$$



Thus, the new coordinates are **$(7, 2)$**.

3.9 Polygon Transformation

Polygon transformation involves applying translation, rotation, reflection, enlargement, and shear to polygons while preserving or modifying their properties.

Key Concepts:

- **Translation:** Moving every vertex of a polygon by a vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$$

- **Rotation:** Rotating a polygon by 90° counterclockwise about the origin transforms each vertex:

$$(x', y') = (-y, x).$$

- **Reflection:** Flipping a polygon across a given axis.

$$\text{Reflection over the y-axis: } (x, y) \to (-x, y).$$

$$\text{Reflection over the x-axis: } (x, y) \to (x, -y).$$

- **Enlargement:** Scaling a polygon by a factor k about the origin:

$$(x', y') = (kx, ky).$$

- **Shear:** Distorting a polygon along an axis while keeping one dimension fixed.

$$\text{Horizontal shear: } (x', y') = (x + ky, y).$$

Examples:

Example 1: Translating a Triangle

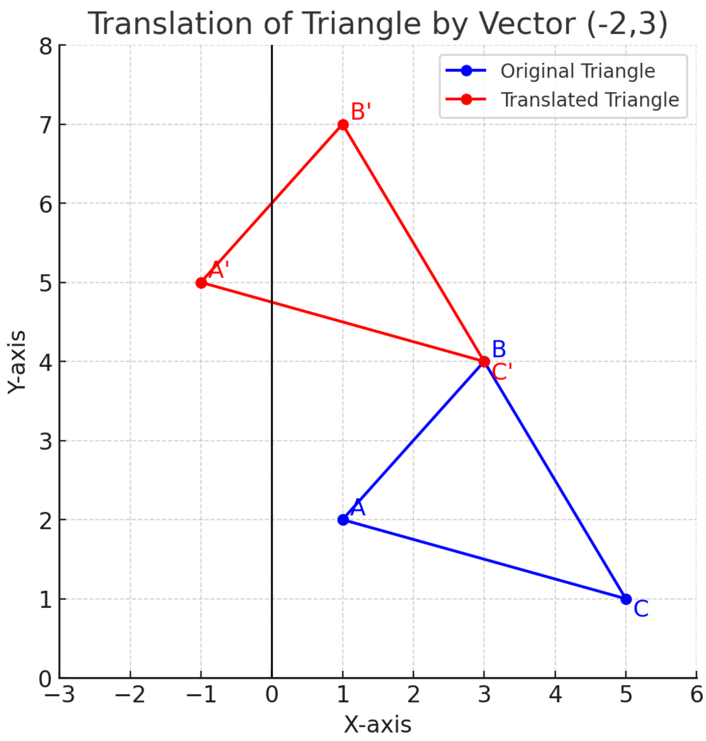
A triangle with vertices $A(1, 2)$, $B(3, 4)$, and $C(5, 1)$ is translated by the vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Find the new coordinates.

Solution:

Using the translation formula:

$$A'(1 - 2, 2 + 3) = (-1, 5),$$

$$B'(3 - 2, 4 + 3) = (1, 7),$$
$$C'(5 - 2, 1 + 3) = (3, 4).$$



Thus, the translated triangle has vertices **$A'(-1,5)$, $B'(1,7)$, and $C'(3,4)$**.

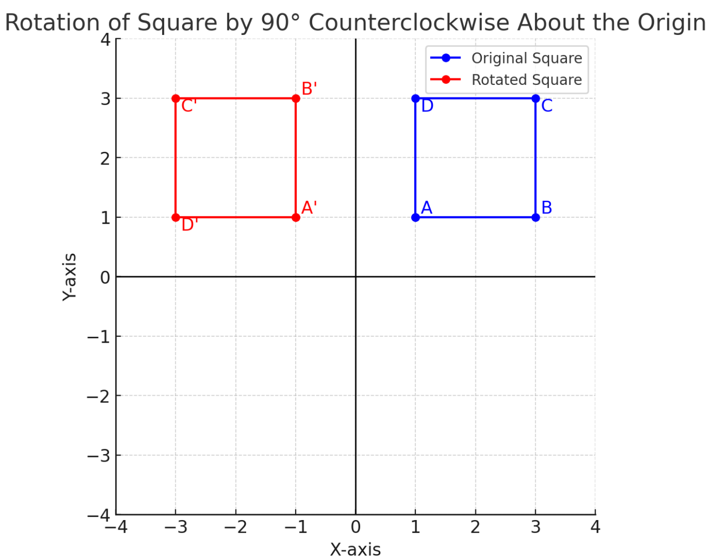
Example 2: Rotating a Square by $90°$
A square has vertices $A(1,1)$, $B(3,1)$, $C(3,3)$, and $D(1,3)$. Find the new coordinates after rotating it $90°$ counterclockwise about the origin.
Solution:
Using the rotation formula $(x', y') = (-y, x)$:

$$A'(1,1) \to (-1,1),$$

$$B'(3,1) \to (-1,3),$$

$$C'(3,3) \to (-3,3),$$

$$D'(1,3) \to (-3,1).$$



Thus, the rotated square has vertices **$A'(-1,1)$, $B'(-1,3)$, $C'(-3,3)$, and $D'(-3,1)$**.

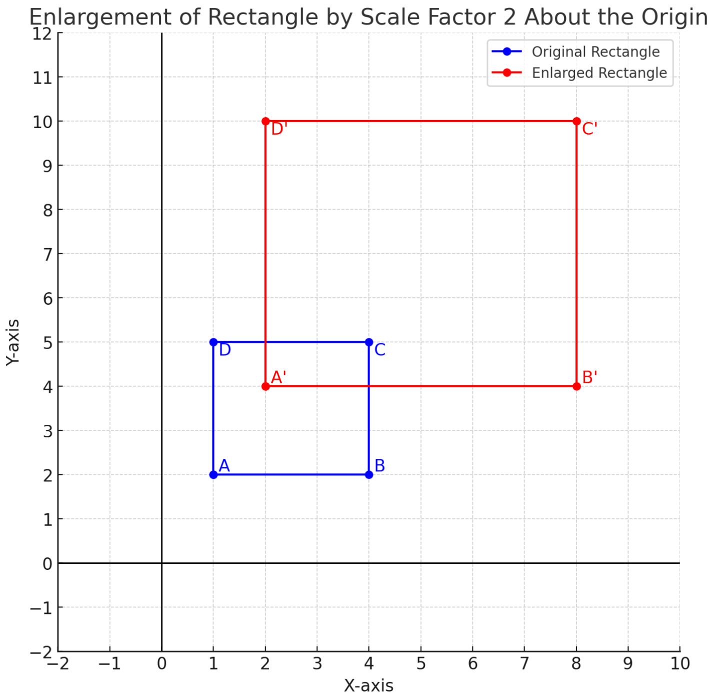
Example 3: Enlarging a Rectangle by Scale Factor 2
A rectangle has vertices $A(1,2)$, $B(4,2)$, $C(4,5)$, and $D(1,5)$. Find the new coordinates after enlarging it by a scale factor of 2 about the origin.
Solution:
Using the enlargement formula $(x', y') = (kx, ky)$:

$$A'(1 \times 2, 2 \times 2) = (2,4),$$

$$B'(4 \times 2, 2 \times 2) = (8,4),$$

$$C'(4 \times 2, 5 \times 2) = (8,10),$$

$$D'(1 \times 2, 5 \times 2) = (2,10).$$



Thus, the enlarged rectangle has vertices **$A'(2,4)$, $B'(8,4)$, $C'(8,10)$, and $D'(2,10)$**.

Example 4: Shearing a Parallelogram
A parallelogram has vertices $A(1,1)$, $B(4,1)$, $C(5,4)$, and $D(2,4)$. It is subjected to a horizontal shear transformation with shear factor $k = 2$ along the x-axis. Find the new coordinates.
Solution:
Using the shear transformation formula:

$$(x', y') = (x + ky, y).$$

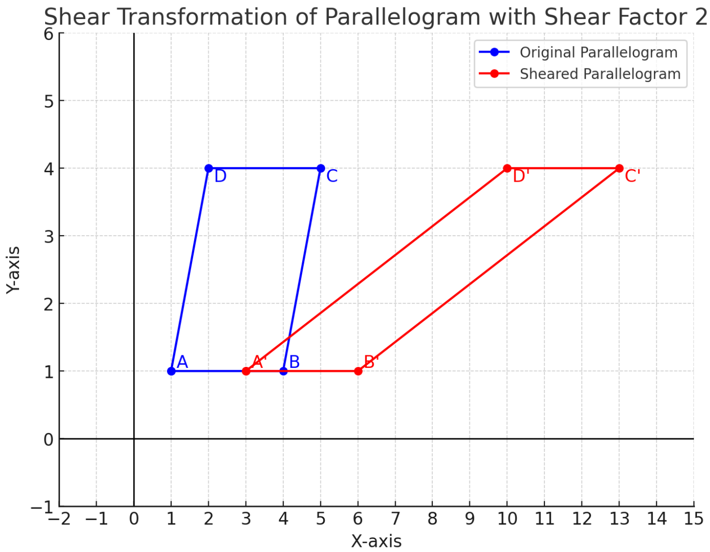
Applying the shear to each vertex:

$$A'(1 + 2(1), 1) = (3,1),$$

$$B'(4 + 2(1), 1) = (6,1),$$

$$C'(5 + 2(4), 4) = (13,4),$$

$$D'(2 + 2(4), 4) = (10,4).$$



Thus, the sheared parallelogram has vertices **$A'(3,1)$, $B'(6,1)$, $C'(13,4)$, and $D'(10,4)$**.

Chapter 4: Trigonometry

4.1 Trigonometric Ratios and identities

Trigonometric ratios relate the angles and sides of right-angled triangles. The three primary trigonometric functions are sine, cosine, and tangent.

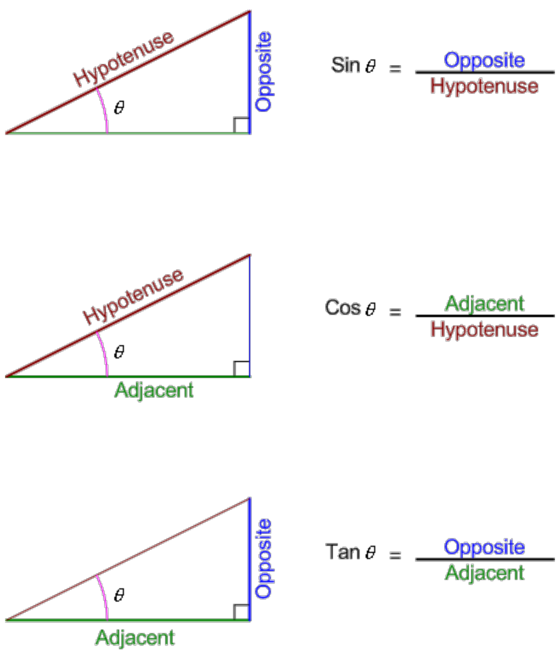
Key Concepts:

- **Right-Angled Triangle Definition:** In a right-angled triangle, the three sides are:
  - Hypotenuse: The longest side, opposite the right angle.
  - Opposite Side: The side opposite to the given angle.
  - Adjacent Side: The side next to the given angle (excluding the hypotenuse).
- **Trigonometric Ratios:** The three basic trigonometric ratios for an angle θ in a right-angled triangle are:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



- **Common Trigonometric Values:**

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

- **Reciprocal Trigonometric Functions:**
  - Cosecant: $\csc\theta = \frac{1}{\sin\theta}$
  - Secant: $\sec\theta = \frac{1}{\cos\theta}$
  - Cotangent: $\cot\theta = \frac{1}{\tan\theta}$
- **Pythagorean Identity:**

$$\sin^2\theta + \cos^2\theta = 1.$$

Examples:

Example 1: Given a right-angled triangle where $\theta = 30^\circ$ and the hypotenuse is 10 cm, find the opposite and adjacent sides.

Solution:

Step 1: Use trigonometric ratios:

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Substituting values:

$$\frac{x}{10} = \frac{1}{2}.$$

Solving for x:

$$x = 10 \times \frac{1}{2} = 5 \text{ cm}.$$

Step 2: Use the cosine function for the adjacent side:

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

$$\frac{y}{10} = \frac{\sqrt{3}}{2}.$$

Solving for y:

$$y = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \text{ cm}.$$

Thus, the opposite side is 5 cm, and the adjacent side is $5\sqrt{3}$ cm.

Example 2: A ladder is leaning against a vertical wall. The foot of the ladder is 4 m from the base of the wall, and the ladder makes an angle of 60° with the ground. Find the length of the ladder.

Solution:

Step 1: Identify the known values: - The adjacent side (distance from the wall) is 4 m. - The hypotenuse is the length of the ladder. - The angle given is 60°. Step 2: Use the cosine function:

$$\cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

$$\frac{4}{h} = \frac{1}{2}.$$

Step 3: Solve for h:

$$h = \frac{4}{\frac{1}{2}} = 8 \text{ m}.$$

Thus, the ladder is 8 m long.

Example 3: Find $\tan\theta$ if $\sin\theta = 0.6$.

Solution:

Step 1: Use the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Substituting $\sin\theta = 0.6$:

$$(0.6)^2 + \cos^2\theta = 1.$$

$$0.36 + \cos^2\theta = 1.$$

$$\cos^2\theta = 0.64.$$

$$\cos\theta = 0.8.$$

Step 2: Use the tangent formula:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

$$\tan\theta = \frac{0.6}{0.8} = 0.75.$$

Thus, $\tan\theta = 0.75$.

Example 4: Given that $\cos x = \frac{12}{13}$, evaluate $\frac{1-\tan x}{\tan x}$.

Solution:

Step 1: Use the Pythagorean identity to find $\sin x$:

$$\sin^2 x + \cos^2 x = 1.$$

Substituting $\cos x = \frac{12}{13}$:

$$\sin^2 x + \left(\frac{12}{13}\right)^2 = 1.$$

$$\sin^2 x + \frac{144}{169} = 1.$$

$$\sin^2 x = 1 - \frac{144}{169} = \frac{169}{169} - \frac{144}{169} = \frac{25}{169}.$$

$$\sin x = \frac{5}{13}.$$

Step 2: Calculate $\tan x$:

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}.$$

Step 3: Evaluate $\frac{1-\tan x}{\tan x}$:

$$\frac{1 - \frac{5}{12}}{\frac{5}{12}}.$$

Simplify the numerator:

$$1 - \frac{5}{12} = \frac{12}{12} - \frac{5}{12} = \frac{7}{12}.$$

Now, divide:

$$\frac{\frac{7}{12}}{\frac{5}{12}} = \frac{7}{12} \times \frac{12}{5} = \frac{7}{5}.$$

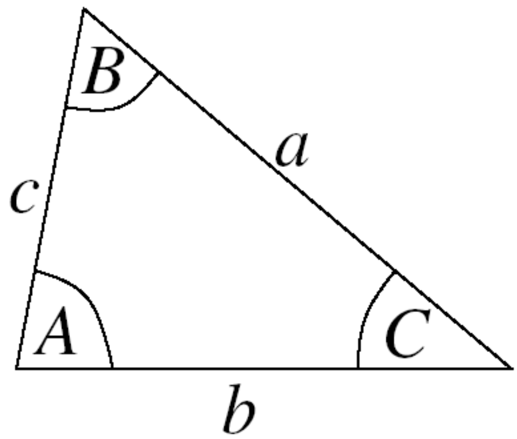
Thus, the final answer is:

$$\frac{7}{5}$$

4.2 Sine and Cosine Rules

The sine and cosine rules are used to find unknown sides and angles in non-right-angled triangles.

Key Concepts:



- **Sine Rule:** Used when given:
  - Two angles and one side (AAS or ASA).
  - Two sides and a non-included angle (SSA).

  The sine rule states:
  $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$
  where a, b, c are the sides opposite to angles A, B, C respectively.

- **Cosine Rule:** Used when given:
  - Two sides and the included angle (SAS).
  - Three sides (SSS) when finding an angle.

  The cosine rule states:
  $$c^2 = a^2 + b^2 - 2ab\cos C.$$
  It can also be used to find angles:
  $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Examples:

Example 1: Find the length of side c in a triangle where $a = 7$ cm, $b = 9$ cm, and $\angle C = 120°$.

Solution:

Step 1: Use the cosine rule:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Substituting values:

$$c^2 = 7^2 + 9^2 - 2(7)(9)\cos 120°.$$

Since $\cos 120° = -\frac{1}{2}$:

$$c^2 = 49 + 81 - 2(7)(9) \times \left(-\frac{1}{2}\right).$$

$$c^2 = 49 + 81 + 63 = 193.$$

$$c = \sqrt{193} \approx 13.9 \text{ cm}.$$

Thus, $c \approx 13.9$ cm.

Example 2: Find angle B in a triangle where $a = 8$ cm, $b = 10$ cm, and $c = 12$ cm.

Solution:

Step 1: Use the cosine rule to find $\cos B$:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

Substituting values:

$$\cos B = \frac{8^2 + 12^2 - 10^2}{2(8)(12)}.$$

$$\cos B = \frac{64 + 144 - 100}{192} = \frac{108}{192}.$$

$$\cos B = 0.5625.$$

Step 2: Find B using inverse cosine:

$$B = \cos^{-1}(0.5625) \approx 55.2°.$$

Thus, $\angle B \approx 55.2°$.

Example 3: Find the missing side a in a triangle where $A = 40°$, $B = 75°$, and $b = 15$ cm.

Solution:

Step 1: Find angle C:

$$C = 180° - A - B = 180° - 40° - 75° = 65°.$$

Step 2: Use the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Substituting values:

$$\frac{a}{\sin 40°} = \frac{15}{\sin 75°}.$$

Solving for a:

$$a = \frac{15 \times \sin 40°}{\sin 75°}.$$

$$a \approx \frac{15 \times 0.6428}{0.9659} = \frac{9.642}{0.9659} \approx 9.99 \text{ cm}.$$

Thus, $a \approx 10.0$ cm.

4.3 Angles of Elevation and Depression, bearing

Angles of elevation and depression are used to describe the inclination or declination of a line of sight relative to the horizontal plane.

Key Concepts:

- **Angle of Elevation:** The angle between the horizontal line and the line of sight when an observer looks up at an object.
- **Angle of Depression:** The angle between the horizontal line and the line of sight when an observer looks down at an object.

Examples:

Example 1: A person standing 50 m away from a tower observes the top of the tower at an angle of elevation of $35°$. Find the height of the tower.

Solution:

Step 1: Identify the known values: - The horizontal distance to the tower is 50 m. - The angle of elevation is 35°. - The height of the tower is the opposite side.
Step 2: Use the tangent function:

$$\tan 35° = \frac{\text{height}}{\text{distance}}.$$

$$\tan 35° = \frac{h}{50}.$$

Step 3: Solve for h:

$$h = 50 \times \tan 35°.$$

Using $\tan 35° \approx 0.7002$:

$$h = 50 \times 0.7002 = 35.01 \text{ m}.$$

Thus, the height of the tower is approximately 35.0 m.

Example 2: A pilot at an altitude of 2 km observes an airport at an angle of depression of $20°$. Find the horizontal distance from the airplane to the airport.

Solution:

Step 1: Identify the known values: - The altitude (vertical height) is 2 km. - The angle of depression is $20°$. - The horizontal distance is the adjacent side. Step 2: Use the tangent function:

$$\tan 20° = \frac{\text{altitude}}{\text{horizontal distance}}.$$

$$\tan 20° = \frac{2}{d}.$$

Step 3: Solve for d:

$$d = \frac{2}{\tan 20°}.$$

Using $\tan 20° \approx 0.3640$:

$$d = \frac{2}{0.3640} \approx 5.49 \text{ km}.$$

Thus, the horizontal distance is approximately 5.49 km.

Example 3: A boat is sailing towards a lighthouse. When the boat is 300 m from the lighthouse, the angle of elevation to the top of the lighthouse is $25°$. If the height of the lighthouse is 60 m, find the distance from the top of the lighthouse to the boat.

Solution:

Step 1: Identify the known values: - The horizontal distance is 300 m. - The height of the lighthouse is 60 m. - The required distance is the hypotenuse. Step 2: Use the sine function:

$$\sin 25° = \frac{60}{d}.$$

Step 3: Solve for d:

$$d = \frac{60}{\sin 25°}.$$

Using $\sin 25° \approx 0.4226$:

$$d = \frac{60}{0.4226} \approx 142.02 \text{ m}.$$

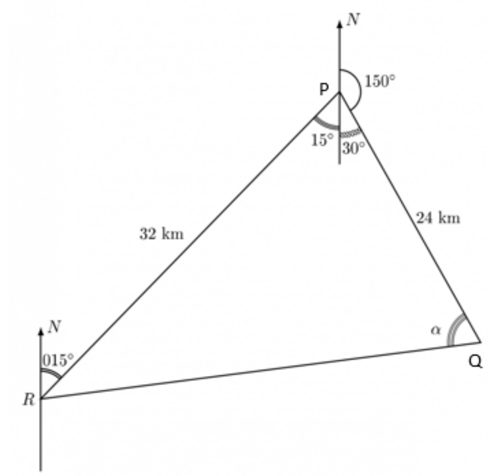
Thus, the distance from the top of the lighthouse to the boat is approximately 142.0 m.

Example 3: The bearing of Q from P is 015° and the bearing of P from R is 015°. If Q and R are 24 km and 32 km respectively from P:

(i) Represent this information in a diagram.

(ii) Calculate the distance between Q and R, correct to two decimal places.

(iii) Find the bearing of R from Q, correct to the nearest degree.



Solution:

Step 1: Represent the given bearings in a diagram. Since both bearings are measured from the north, we construct a diagram accordingly.
Step 2: Use the cosine rule to find the distance between Q and R. The angle between PQ and PR is:

$$\theta = (15^\circ + 35^\circ) = 145^\circ.$$

Using the cosine rule:

$$|QR|^2 = 32^2 + 24^2 - 2 \times 32 \times 24 \times \cos 45^\circ$$

$$|QR|^2 = 1024 + 576 - 1536 \cos 45^\circ$$

$$= 1600 - 1086.1056$$

$$|QR|^2 = 513.8944$$

$$|QR| = \sqrt{513.8944} = 22.669 \text{ km}$$

$$\approx 22.67 \text{ km to 2 dp}$$

Step 3: Find the angle ∠PQR using the sine rule:

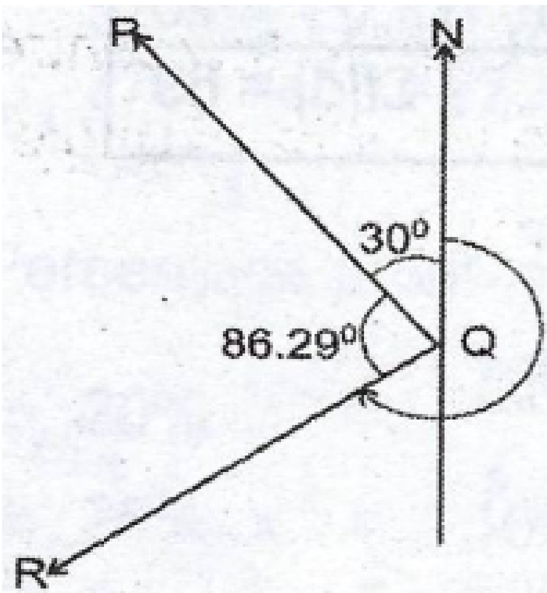
$$\frac{32}{\sin \angle PQR} = \frac{22.67}{\sin 45°}$$

$$\sin \angle PQR = \frac{32 \times \sin 45°}{22.67}$$

$$= 0.9981$$

$$\angle PQR = \sin^{-1}(0.9981) = 86.4787°$$

Step 4: Calculate the bearing of R from Q:



The bearing of R from Q is given by the reflex angle NQR. Thus,

$$\text{Reflex } \angle NQR = 360° - (86.47° + 30°)$$

$$= 360^\circ - 116.47^\circ$$

$$= 243.53^\circ$$

Hence, the bearing of R from Q:

$$\approx 244^\circ \quad \text{(to the nearest degree)}$$

4.4 Circular Measure

Circular measure involves working with angles in both degrees and radians, as well as calculating arc lengths and sector areas.

Key Concepts:

- **Radians and Degrees:** - A radian is an alternative unit for measuring angles, where one full revolution ($360°$) equals 2π radians. - The conversion formulas between degrees and radians are:

$$1° = \frac{\pi}{180} \text{ radians}, \quad 1 \text{ radian} = \frac{180}{\pi}^{\circ}.$$

- **Arc Length:** - The length of an arc of a circle is given by:

$$l = r\theta,$$

  where r is the radius and θ is the central angle in radians.

- **Sector Area:** - The area of a sector of a circle is given by:

$$A = \frac{1}{2}r^2\theta,$$

  where r is the radius and θ is the central angle in radians.

- **Applications:** - Circular measure is used in physics, engineering, and navigation for measuring distances along circular paths.

Examples:

Example 1: Convert $135°$ to radians.

Solution:

Using the conversion formula:

$$\theta = 135° \times \frac{\pi}{180}.$$

$$\theta = \frac{135\pi}{180}.$$

$$\theta = \frac{3\pi}{4} \text{ radians}.$$

Thus, $135° = \frac{3\pi}{4}$ radians.

Example 2: Convert 2.5 radians to degrees.

Solution:

Using the conversion formula:

$$\theta = 2.5 \times \frac{180}{\pi}.$$

$$\theta = \frac{450}{\pi} \approx 143.24^\circ.$$

Thus, 2.5 radians $\approx 143.2^\circ$.

Example 3: Find the length of an arc in a circle of radius 10 cm subtended by a central angle of 1.2 radians.

Solution:

Using the arc length formula:

$$l = r\theta.$$

Substituting values:

$$l = 10 \times 1.2 = 12 \text{ cm}.$$

Thus, the arc length is 12 cm.

Example 4: Find the area of a sector with radius 8 cm and central angle 2 radians.

Solution:

Using the sector area formula:

$$A = \frac{1}{2}r^2\theta.$$

Substituting values:

$$A = \frac{1}{2} \times 8^2 \times 2.$$

$$A = \frac{1}{2} \times 64 \times 2 = 64 \text{ cm}^2.$$

Thus, the area of the sector is 64 cm**²**.

Chapter 5: Statistics and Probability

5.1 Data Representation

Data representation is crucial in statistics for organizing and analyzing information. Various methods are used to display and interpret data effectively.

Key Concepts:

- **Tables:** Data can be organized in tables for better readability and analysis.
- **Bar Charts:** Used to represent categorical data with rectangular bars. Horizontal axis of a bar chart can be numberical or descriptive.
- **Pie Charts:** Circular charts divided into sectors, where each sector represents a proportion of the whole.
- **Histograms:** Graphical representation of grouped data with bars whose areas represent frequency. Although visually similar to bar charts histograms visualize quantitative data or numerical data, whereas bar charts display categorical variables. Veritably axis of histogram represent frequency while horizontal some numerical data. (see Example 2).
- **Frequency Distributions:** Tables that display the frequency of different outcomes in a dataset.
- **Cumulative Frequency Tables:** Tables that show the accumulation of frequencies up to a given class interval.
- **Cumulative Frequency Curve (Ogive):** A graphical representation of cumulative frequency that helps estimate median, percentiles, and quartiles.

Examples:

Example 1: Construct a frequency distribution table from the given data:
$\{3, 5, 7, 3, 5, 7, 9, 7, 5, 3, 7, 5, 9, 7, 5\}$

Solution:

Value (x)	Frequency (f)
3	3
5	4
7	5
9	2

The frequency table summarizes the given dataset.

Example 2: The table below shows the marks obtained by students in a test. Construct a histogram.

Marks Range	Frequency
0-10	2
10-20	4
20-30	7
30-40	10
40-50	6

Solution:

A histogram is a bar graph where the class intervals are plotted on the x-axis, and the frequencies are represented by the heights of the bars.

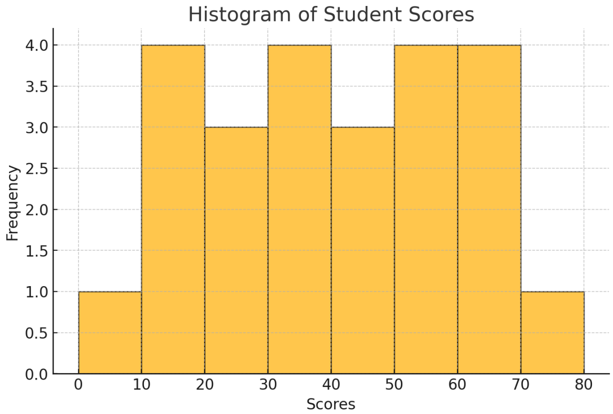


Figure 1: Histogram of student marks.

Note that vertical axis represent frequency (always) while horizontal axis represents some numerical data (in this case students score).

Example 3: The pie chart below represents the distribution of students in a school. If the total number of students is 600, find the number of students in each category.

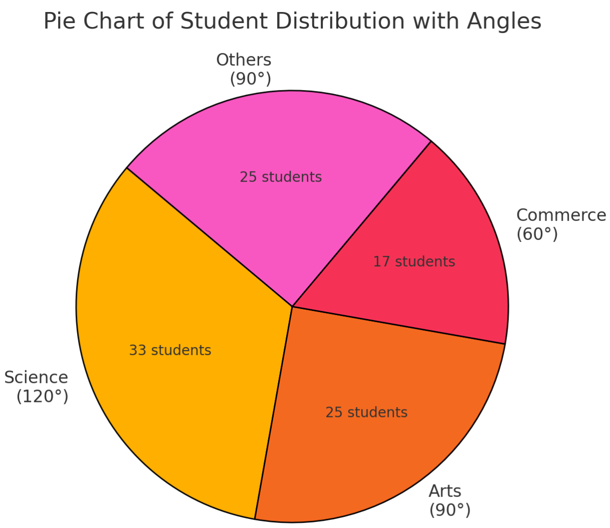


Figure 2: Pie chart representing student distribution.

Solution:

Let the angles of the pie chart sectors be: - Science: 120° - Arts: 90° - Commerce: 60° - Others: 90°
The total angle in a pie chart is 360°, so we calculate each category as:

$$\text{Number of students} = \frac{\text{sector angle}}{360^\circ} \times \text{total students}.$$

$$\text{Science} = \frac{120}{360} \times 600 = 200.$$

$$\text{Arts} = \frac{90}{360} \times 600 = 150.$$

$$\text{Commerce} = \frac{60}{360} \times 600 = 100.$$

$$\text{Others} = \frac{90}{360} \times 600 = 150.$$

Thus, the number of students in each category is: Science = 200, Arts = 150, Commerce = 100, Others = 150.

Example 5: Construct a cumulative frequency table and draw the cumulative frequency curve (ogive) for the following data.

Class Interval	Frequency
0-10	3
10-20	7
20-30	12
30-40	18
40-50	10

Solution:

Step 1: Construct the cumulative frequency table by adding up the frequencies progressively.

Class Interval	Frequency	Cumulative Frequency
0-10	3	3
10-20	7	3 + 7 = 10
20-30	12	10 + 12 = 22
30-40	18	22 + 18 = 40
40-50	10	40 + 10 = 50

Step 2: Plot the cumulative frequency curve (ogive).

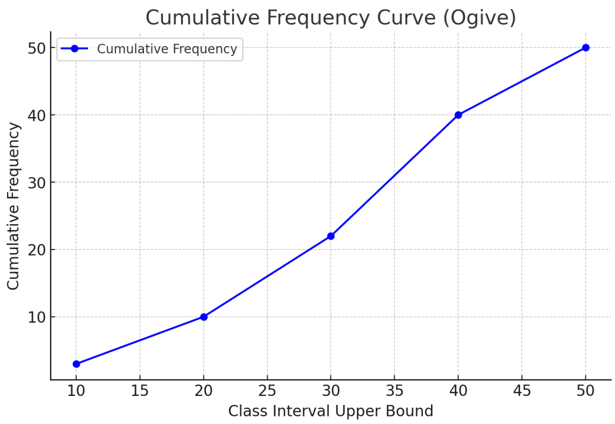


Figure 3: Cumulative Frequency Curve (Ogive)

Step 3: Use the ogive to estimate the median and percentiles.
- The median is the value corresponding to $\frac{n}{2} = \frac{50}{2} = 25$ on the cumulative frequency axis. - The 25th percentile is obtained at $\frac{50}{4} = 12.5$ on the cumulative frequency axis. - The 75th percentile is obtained at $\frac{3\times 50}{4} = 37.5$ on the cumulative frequency axis.
By tracing these values on the ogive, we estimate:

$$\text{Median} \approx 27, \quad 25^{\text{th}} \text{ percentile} \approx 18, \quad 75^{\text{th}} \text{ percentile} \approx 35.$$

Thus, the cumulative frequency curve helps in estimating key statistical measures.

5.2 Measures of Central Tendency

Measures of central tendency describe the center of a data set using three main statistics: mean, median, and mode.

Key Concepts:

- **Mean:** The arithmetic average of a data set, calculated as:

$$\text{Mean} = \frac{\sum xf}{\sum f},$$

  where x represents the class midpoint, and f is the frequency.
  In case of grouped data use midpoints of each group to estimate the mean (see Example 2.)

- **Median:** The middle value in an ordered data set. Note that the data must be ordered first.

  - For an odd number of values: The median is the middle number.
    Example: Data $= 1, 2, 3, 4, 5$
    Median $= 3$.
  - For an even number of values: The median is the average of the two middle numbers.
    Example: Data $= 1, 2, 3, 4, 5, 6$
    Median $= \frac{3+4}{2} = 3.5$.

- **Mode:** The most frequently occurring value in a dataset. For grouped data, the modal class is the class with the highest frequency.

  - If a number appears most frequently, it is the mode.
    Example: Data $= 1, 2, 2, 3, 2, 4, 5, 6, 4, 3, 2, 3, 4, 2, 3, 2, 6, 5$
    Frequency:
    * 1 appears once.
    * 2 appears **6** times.
    * 3 appears 5 times.
    * 4 appears 3 times.
    * 5 appears 2 times.
    * 6 appears 2 times.

    Mode = **2** (most frequent value).

- **Graph work:** You can find certain measures of central tendency from cumulative frequency graph or histograms. See examples 3 and 4 for details.

Examples:

Example 1: Find the Mean, Median, and Mode of the Given Data Set.
Given Data:
$12, 7, 10, 15, 10, 18, 12, 10, 14, 7, 16, 12, 10, 9, 10$

Solution:
Step 1: Find the Mode

- Count the frequency of each number:
  - 7 appears twice.
  - 9 appears once.
  - 10 appears **5** times.
  - 12 appears 3 times.
  - 14 appears once.
  - 15 appears once.
  - 16 appears once.
  - 18 appears once.
- The mode is the number that appears most frequently.

$$\text{Mode} = 10$$

Step 2: Find the Median

- Arrange the data in ascending order:

$$7, 7, 9, 10, 10, 10, 10, 10, 12, 12, 12, 14, 15, 16, 18$$

- Find the middle value. Since there are 15 numbers (odd count), the median is the 8th value.

$$\text{Median} = 10$$

Step 3: Find the Mean

- Use the mean formula:

$$\text{Mean} = \frac{\sum x}{n}$$

- Calculate the sum of all numbers:

$$7+7+9+10+10+10+10+10+12+12+12+14+15+16+18 = 162$$

- Divide by the total number of values ($n = 15$):

Example 2: Find the mean from the following frequency distribution.

Class Interval	Frequency (f)	Midpoint (x)
0 - 10	4	5
10 - 20	6	15
20 - 30	8	25
30 - 40	10	35
40 - 50	7	45

Solution:

Using the formula:

$$\text{Mean} = \frac{\sum xf}{\sum f}.$$

$$\text{Mean} = \frac{(5 \times 4) + (15 \times 6) + (25 \times 8) + (35 \times 10) + (45 \times 7)}{4 + 6 + 8 + 10 + 7}.$$

$$= \frac{20 + 90 + 200 + 350 + 315}{35}.$$

$$= \frac{975}{35} = 27.86.$$

Thus, the mean is approximately 27.86.

Example 3: Determine the median from the cumulative frequency curve (ogive) below.

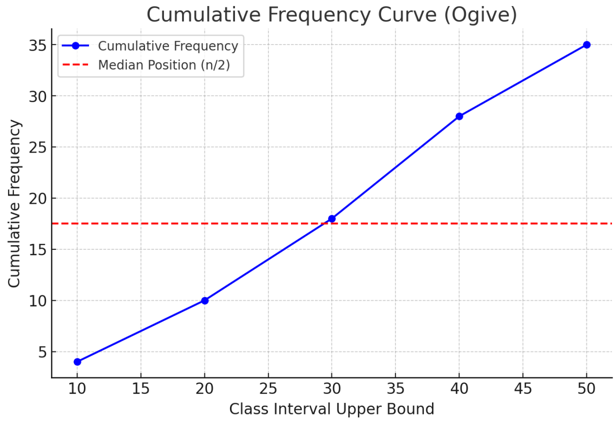


Figure 4: Cumulative Frequency Curve (Ogive)

Solution:

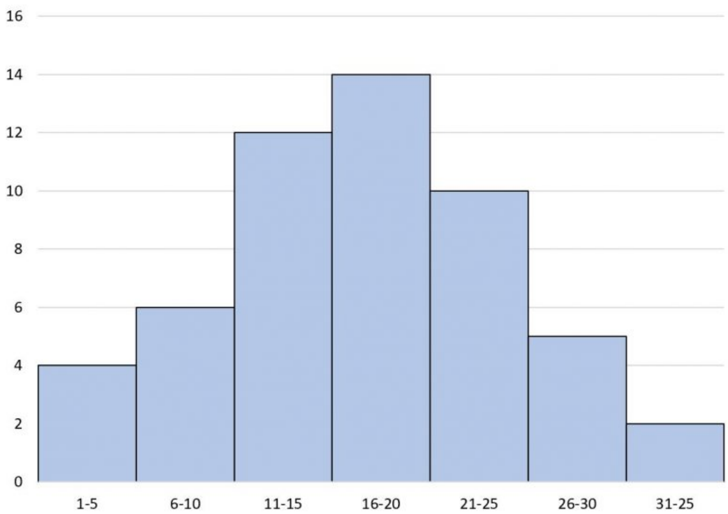
Step 1: Identify the total number of observations n.

Step 2: Find the median position: $\frac{n}{2} = \frac{35}{2} = 17.5$.

Step 3: Locate the corresponding value on the cumulative frequency curve: dashed red line on the graph.

Step 4: Use interpolation if necessary to estimate the median value. Solution is the intersection of cummulative frequency curve and dashed line. From the graph, the estimated median is around 28.

Example 4: Determine the Mode from the Given Histogram.



Solution:

Step 1: Identify the modal class.

- The modal class is the class interval with the highest frequency, which is $16 - 20$.

Step 2: Use the histogram and the line intersection method.

- Draw two diagonal lines from the tops of the bars adjacent to the modal class, forming an "X" shape.

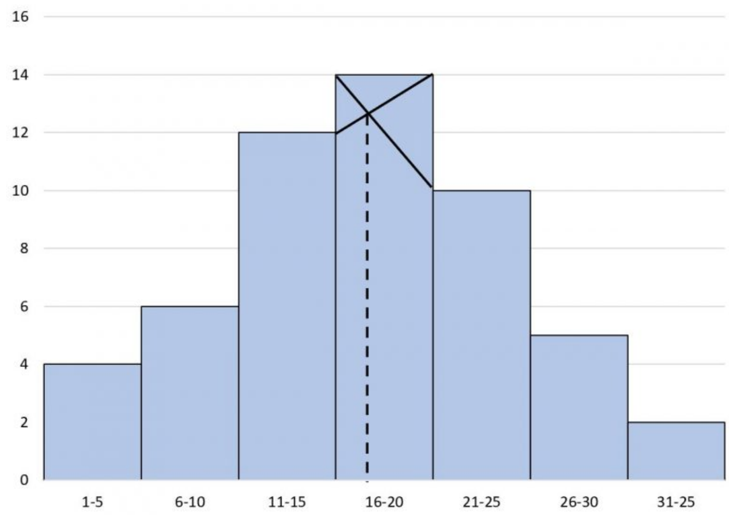


Figure 5: Histogram with Mode Estimation Using Line Intersection.

- The intersection of these lines falls near $x \approx 17$, which represents the estimated mode.

Step 3: Conclusion.

- From the visual estimation, the mode is approximately:

$$\textbf{Mode} \approx 17.$$

Thus, the mode is visually estimated to be around 17.

5.3 Measures of Dispersion

Measures of dispersion describe how spread out the data is. The key measures include range, quartiles, variance, mean deviation, and standard deviation.

Key Concepts:

- **Range:** The difference between the maximum and minimum values in a dataset.

$$\text{Range} = \max(x) - \min(x)$$

- **Quantiles:** Values that divide a data set into equal parts. Common quantiles include:
  * **Quartiles:** Divide the data into four equal parts.
  * **Median (Q2):** The middle value of an ordered data set.
  * **Lower Quartile (Q1):** The median of the lower half of data.
  * **Upper Quartile (Q3):** The median of the upper half of data.
  * **Interquartile Range (IQR):** Measures the spread of the middle 50

    $$\text{IQR} = Q3 - Q1$$

  * **Semi-Interquartile Range:** Half of the interquartile range.

    $$\text{SIQR} = \frac{Q3 - Q1}{2}$$

- **Mean Deviation:** The average absolute deviation of each data point from the mean.

$$\text{Mean Deviation} = \frac{\sum |x - \bar{x}|}{n}$$

- **Variance:** Measures the average squared deviation from the mean.

$$\sigma^2 = \frac{\sum (x - \bar{x})^2}{n}$$

- **Standard Deviation:** The square root of variance, showing the spread of data.

$$\sigma = \sqrt{\sigma^2}$$

Examples:

Example 1: Find the range, quartiles, and interquartile range for the following data set.

Given Data:
$4, 8, 15, 16, 23, 42, 50, 55, 60$

Solution:

Step 1: Find the Range

$$\text{Range} = 60 - 4 = 56.$$

Step 2: Find the Quartiles

- Median (Q2): The middle value is 23.
- Lower Quartile (Q1): The median of the lower half $(4, 8, 15, 16)$ is:
$$Q1 = \frac{8 + 15}{2} = 11.5.$$
- Upper Quartile (Q3): The median of the upper half $(42, 50, 55, 60)$ is:
$$Q3 = \frac{50 + 55}{2} = 52.5.$$

Step 3: Calculate Interquartile Range (IQR)

$$\text{IQR} = Q3 - Q1 = 52.5 - 11.5 = 41.$$

Step 4: Compute Semi-Interquartile Range

$$\text{SIQR} = \frac{IQR}{2} = \frac{41}{2} = 20.5.$$

Thus, the range is **56**, IQR is **41**, and SIQR is **20.5**.

Example 2: Find the Mean Deviation, Variance, and Standard Deviation for the following data set.

Given Data:
$5, 10, 15, 20, 25$

Solution:

Step 1: Compute the Mean

$$\bar{x} = \frac{5 + 10 + 15 + 20 + 25}{5} = \frac{75}{5} = 15.$$

Step 2: Compute Mean Deviation

$$\sum |x - \bar{x}| = |5 - 15| + |10 - 15| + |15 - 15| + |20 - 15| + |25 - 15|$$

$$= 10 + 5 + 0 + 5 + 10 = 30.$$

$$\text{Mean Deviation} = \frac{30}{5} = 6.$$

Step 3: Compute Variance

$$\sigma^2 = \frac{\sum (x - \bar{x})^2}{n}$$

$$= \frac{(5 - 15)^2 + (10 - 15)^2 + (15 - 15)^2 + (20 - 15)^2 + (25 - 15)^2}{5}$$

$$= \frac{100 + 25 + 0 + 25 + 100}{5} = \frac{250}{5} = 50.$$

Step 4: Compute Standard Deviation

$$\sigma = \sqrt{50} \approx 7.07.$$

Thus, the mean deviation is **6**, variance is **50**, and standard deviation is **7.07**.

Chapter 6: Probability

6.1 Simple Probability

Probability is the measure of how likely an event is to occur. It is expressed as a number between 0 (impossible) and 1 (certain).

Key Concepts:

- **Experiment:** A process that leads to an outcome.
- **Sample Space (S):** The set of all possible outcomes of an experiment.
- **Event (E):** A subset of the sample space representing specific outcomes.
- **Outcome:** A single possible result of an experiment.
  - Example: Rolling a die can result in outcomes $1, 2, 3, 4, 5$, or 6.
  - Example: Flipping a coin has two possible outcomes: **Heads** or **Tails**.
- **Outcome Space (Sample Space, S):** The set of all possible outcomes of an experiment.
  - Example: The sample space for rolling a six-sided die is:
  $$S = \{1, 2, 3, 4, 5, 6\}.$$
  - Example: The sample space for flipping two coins is:
  $$S = \{\text{HH, HT, TH, TT}\}.$$
- **Probability of an Event:** The probability of an event occurring is given by:
$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}.$$
- **Complement of an Event:** The probability of an event not occurring is:
$$P(E') = 1 - P(E).$$

Examples:

Example 1: Finding the Probability of Rolling a Die
Solution:
Step 1: Identify the sample space.
The sample space for rolling a die is:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Step 2: Find the probability of rolling a 4.

$$P(4) = \frac{1}{6}.$$

Thus, the probability of rolling a 4 is **$\frac{1}{6}$**.

Example 2: Probability of Drawing a Red Card from a Deck of 52 Cards
Solution:
Step 1: Identify the total outcomes.
A standard deck has 52 cards, and half (26) are red.
Step 2: Compute the probability.

$$P(\text{Red Card}) = \frac{26}{52} = \frac{1}{2}.$$

Thus, the probability of drawing a red card is $**\frac{1}{2}**$.

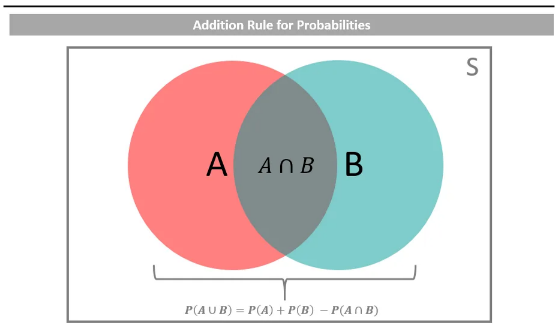
6.2 Combination of Events

In probability, events can be combined using the **addition law** (for "or" situations) and the **multiplication law** (for "and" situations). Events can also be classified as **mutually exclusive** or **independent**.

Key Concepts:

- **Addition Law of Probability:** Used for the probability of either event A or event B occurring.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



- **Mutually Exclusive Events:** Two events that cannot happen at the same time.

$$P(A \cap B) = 0.$$

  If A and B are mutually exclusive, then:

$$P(A \cup B) = P(A) + P(B).$$

- **Independent Events:** The occurrence of one event does not affect the probability of the other.

$$P(A \cap B) = P(A) \times P(B).$$

Examples:

Example 1: Addition Rule with Overlapping Events
A deck of 52 cards contains 13 hearts and 13 face cards. If a card is drawn at random, what is the probability that it is a heart or a face card?
Solution:
Step 1: Use the addition formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Step 2: Find the probabilities:

$$P(\text{Heart}) = \frac{13}{52}, \quad P(\text{Face Card}) = \frac{12}{52}.$$

Since there are 3 face cards in hearts,

$$P(\text{Heart} \cap \text{Face Card}) = \frac{3}{52}.$$

Step 3: Compute:

$$P(\text{Heart or Face Card}) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}.$$

Example 2: Probability of Independent Events
A fair die is rolled, and a coin is flipped. What is the probability of getting a 4 on the die and heads on the coin?
Solution:
Step 1: Identify the probabilities:

$$P(\text{Rolling a 4}) = \frac{1}{6}, \quad P(\text{Flipping Heads}) = \frac{1}{2}.$$

Step 2: Since these are independent events, use the multiplication rule:

$$P(4 \cap H) = P(4) \times P(H) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}.$$

Example 3: Mutually Exclusive Events
A bag contains 5 red marbles and 7 blue marbles. If one marble is drawn at random, what is the probability that it is either red or blue?
Solution:
Since drawing a red marble and drawing a blue marble are mutually exclusive events, we use:

$$P(A \cup B) = P(A) + P(B).$$

$$P(\text{Red}) = \frac{5}{12}, \quad P(\text{Blue}) = \frac{7}{12}.$$

$$P(\text{Red or Blue}) = \frac{5}{12} + \frac{7}{12} = 1.$$

Since one of these must occur, the probability is 1.

Example 4: Probability of Winning Independent Contracts
A building contractor tendered for two independent contracts, X and Y. The probabilities that he will win contract X is 0.5 and not win contract Y is 0.3. What is the probability that he will:

1. Win both contracts?
2. Win exactly one of the contracts?
3. Win neither of the contracts?

Solution:
Let:

$$P(X) = 0.5, \quad P(\text{Not } Y) = 0.3.$$

Since $P(\text{Not } Y) = 0.3$, then:

$$P(Y) = 1 - 0.3 = 0.7.$$

(i) Probability of Winning Both Contracts:
Since the events are independent:

$$P(X \cap Y) = P(X) \times P(Y) = 0.5 \times 0.7 = 0.35.$$

(ii) Probability of Winning Exactly One Contract:
This occurs in two ways: - Wins X and loses Y:
$P(X \cap \text{Not } Y) = P(X) \times P(\text{Not } Y)$. - Loses X and wins Y:
$P(\text{Not } X \cap Y) = P(\text{Not } X) \times P(Y)$.
Since $P(\text{Not } X) = 1 - 0.5 = 0.5$, we compute:

$$P(X \cap \text{Not } Y) = 0.5 \times 0.3 = 0.15.$$

$$P(\text{Not } X \cap Y) = 0.5 \times 0.7 = 0.35.$$

Adding both cases:

$$P(\text{Exactly one contract}) = 0.15 + 0.35 = 0.50.$$

(iii) Probability of Winning Neither Contract:

$$P(\text{Not } X \cap \text{Not } Y) = P(\text{Not } X) \times P(\text{Not } Y).$$

$$= 0.5 \times 0.3 = 0.15.$$

Final Answers:

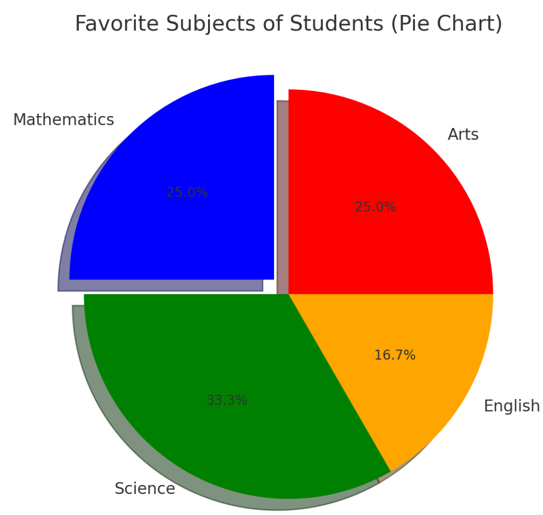
- Probability of winning both contracts: 0.35.
- Probability of winning exactly one contract: 0.50.
- Probability of winning neither contract: 0.15.

6.3 Probability Calculations from Various Graphs

Probability can be estimated from different types of graphical representations, including pie charts, histograms, cumulative frequency graphs, and Venn diagrams.

Examples:

Example 1: Probability from a Pie Chart



A pie chart represents the distribution of students in a school by their favorite subjects: Mathematics (90°), Science (120°), English (60°), and Arts (90°). What is the probability that a randomly chosen student prefers Science?
Solution:
Step 1: Compute the proportion of students who prefer Science.

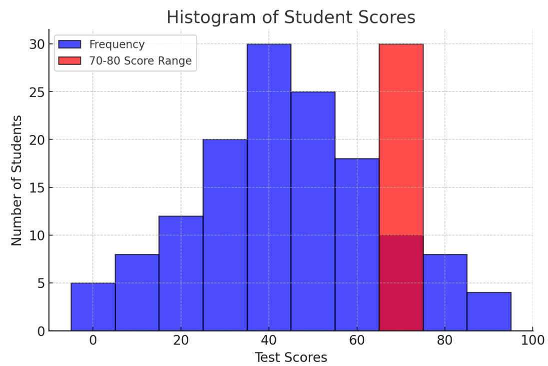
$$\text{Fraction of Science} = \frac{120^\circ}{360^\circ} = \frac{1}{3}.$$

Step 2: Compute probability.

$$P(\text{Science}) = \frac{1}{3} = 0.333.$$

Thus, the probability that a randomly chosen student prefers Science is **0.333 or 33.3%**.

Example 2: Probability from a Histogram



A histogram represents the number of students who scored different marks in a test. The total number of students is 100. Find the probability that a randomly chosen student scored between 70 and 80.

Solution:

Step 1: Use the probability formula.

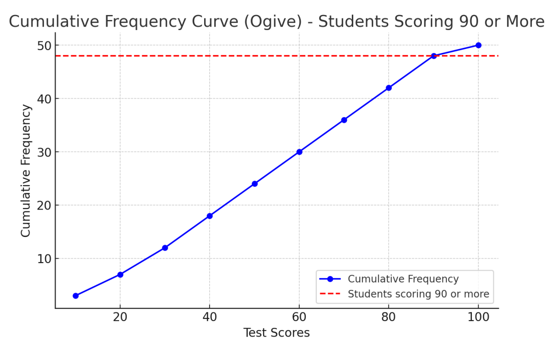
$$P(70 \leq \text{Score} \leq 80) = \frac{\text{Number of students in range}}{\text{Total students}}.$$

$$= \frac{30}{100} = 0.30.$$

Thus, the probability is **0.30 or 30%**.

Example 3: Probability from a Cumulative Frequency Curve

A cumulative frequency graph shows the number of students scoring below certain marks in a test. Find the probability for the student to get distinction for the test if distinction score is 90.



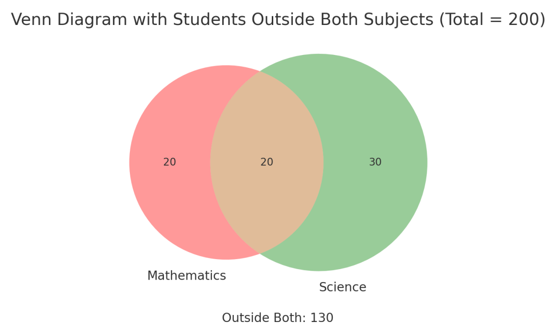
Solution:

Step 1: Compute probability.

$$P(distinction) = P(\text{Score} > 90) = \frac{2}{50} = 0.04.$$

Thus, the probability is **0.04 or 4%**.

Example 4: Probability from a Venn Diagram



A Venn diagram shows the distribution of students who study Mathematics (40 students) and Science (50 students), with 20 students studying both subjects. Total number of studnets is 200. If a student is randomly selected, find
a)the probability that the student studies either Mathematics or Science.
b)the probability the study math only

Part A Solution:

Step 1: Use the addition rule.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$$P(\text{Math or Science}) = \frac{40}{200} + \frac{50}{200} - \frac{20}{100}.$$

$$= \frac{70}{200} = 0.35.$$

Thus, the probability is **0.35 or 35%**.

Part B Solution:

$$P(\text{Math Only}) = \frac{20}{200} = 0.1.$$

Thus, the probability is **0.10 or 10%**.

Chapter 7: Commercial Mathematics

7.1 Profit, Loss, and Discount

Commercial mathematics is essential in business transactions involving cost, revenue, and profit calculations.

Key Concepts:

- **Cost Price (CP):** The original price at which an item is purchased.
- **Selling Price (SP):** The price at which an item is sold.
- **Profit:** The gain obtained when an item is sold for more than its cost price.

$$\text{Profit} = \text{SP} - \text{CP}$$

- **Loss:** The amount lost when an item is sold for less than its cost price.

$$\text{Loss} = \text{CP} - \text{SP}$$

- **Profit Percentage:**

$$\text{Profit }\% = \left(\frac{\text{Profit}}{\text{CP}}\right) \times 100$$

- **Loss Percentage:**

$$\text{Loss }\% = \left(\frac{\text{Loss}}{\text{CP}}\right) \times 100$$

- **Discount:** A reduction in the marked price of an item.

$$\text{Discount} = \text{Marked Price} - \text{Selling Price}$$

- **Discount Percentage:**

$$\text{Discount }\% = \left(\frac{\text{Discount}}{\text{Marked Price}}\right) \times 100$$

- **Markup:** The percentage increase on the cost price before selling.

$$\text{Selling Price} = \text{Cost Price} + \left(\frac{\text{Markup }\% \times \text{CP}}{100}\right)$$

Examples:

Example 1: Finding Profit Percentage
A trader buys a shirt for \$40 and sells it for \$50. Find the profit percentage.
Solution:

Step 1: Compute the profit.

$$\text{Profit} = 50 - 40 = 10.$$

Step 2: Compute the profit percentage.

$$\text{Profit }\% = \left(\frac{10}{40}\right) \times 100 = 25\%.$$

Thus, the profit percentage is **25%**.

Example 2: Finding Discount Percentage
A store sells a bicycle originally marked at $200 for $170 after applying a discount. Find the discount percentage.
Solution:
Step 1: Compute the discount.

$$\text{Discount} = 200 - 170 = 30.$$

Step 2: Compute the discount percentage.

$$\text{Discount }\% = \left(\frac{30}{200}\right) \times 100 = 15\%.$$

Thus, the discount percentage is **15%**.

Example 3: Calculating Actual Percentage Profit
A television set was marked for sale at GH760.00 in order to make a profit of 20%. The television set was actually sold at a discount of 5%. Calculate, correct to 2 significant figures, the actual percentage profit.
Solution:
Step 1: Find the cost price (CP).
Since the marked price includes a 20% profit on the cost price, we set up the equation:

$$\text{Marked Price} = \text{CP} + 20\% \times \text{CP} = 1.2 \times \text{CP}$$

Given the marked price is GH760:

$$\text{CP} = \frac{760}{1.2} = 633.33$$

Step 2: Find the actual selling price (SP).
Since a 5% discount is applied:

$$\text{SP} = \text{Marked Price} - 5\% \times \text{Marked Price}$$

$$\text{SP} = 760 - (0.05 \times 760) = 760 - 38 = 722$$

Step 3: Calculate the actual profit.

$$\text{Profit} = \text{SP} - \text{CP} = 722 - 633.33 = 88.67$$

Step 4: Find the actual profit percentage.

$$\text{Profit \%} = \left(\frac{\text{Profit}}{\text{CP}}\right) \times 100 = \left(\frac{88.67}{633.33}\right) \times 100$$

$$= 14.0\%$$

Thus, the actual percentage profit is **14%** (correct to 2 significant figures).

7.2 Simple and Compound Interest

Interest is the extra amount paid on a loan or earned on savings over time. There are two main types of interest calculations: simple interest and compound interest.

Key Concepts:

- **Simple Interest:** Interest calculated on the original principal amount.

$$I = P \times r \times t$$

  where:

  - I = Interest
  - P = Principal (initial amount)
  - r = Annual interest rate (as a decimal)
  - t = Time in years

- **Total Amount with Simple Interest:**

$$A = P + I = P(1 + rt)$$

- **Compound Interest:** Interest calculated on both the principal and previously earned interest.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

  where:

  - A = Final amount after interest
  - P = Principal
  - r = Annual interest rate (as a decimal)
  - n = Number of times interest is applied per year
  - t = Time in years

- **Continuous Compound Interest:**

$$A = Pe^{rt}$$

  where e is the mathematical constant ≈ 2.718.

Examples:

Example 1: Simple Interest Calculation
John deposits \$5000 in a bank account that pays 4% simple interest per year. Find the total interest earned after 3 years.
Solution:

Step 1: Use the simple interest formula.

$$I = 5000 \times 0.04 \times 3$$

$$I = 600$$

Step 2: Find the total amount.

$$A = P + I = 5000 + 600 = 5600$$

Thus, John earns **$600** in interest, and his total amount after 3 years is **$5600**.

Example 2: Compound Interest Calculation
Sarah invests $2000 in a savings account that offers an annual interest rate of 5%, compounded yearly. Find the total amount after 4 years.
Solution:
Step 1: Use the compound interest formula.

$$A = 2000\left(1 + \frac{0.05}{1}\right)^{1\times 4}$$

$$A = 2000\,(1.05)^4$$

$$A = 2000 \times 1.21550625$$

$$A = 2431.01$$

Thus, Sarah's total amount after 4 years is **$2431.01**.

7.3 Depreciation and Appreciation

Depreciation and appreciation refer to the decrease or increase in the value of an asset over time.

Key Concepts:

- **Depreciation:** The reduction in the value of an asset over time, often due to wear and tear or obsolescence.

$$A = P(1 - r)^t$$

  where:

  - A = Value of the asset after time t
  - P = Initial value (purchase price)
  - r = Rate of depreciation (as a decimal)
  - t = Time in years

- **Appreciation:** The increase in the value of an asset over time, often due to demand or inflation.

$$A = P(1 + r)^t$$

  where:

  - A = Value of the asset after time t
  - P = Initial value
  - r = Rate of appreciation (as a decimal)
  - t = Time in years

Examples:

Example 1: Depreciation Calculation
A car was bought for $15,000 and depreciates at a rate of 10% per year. Find its value after 5 years.
Solution:
Step 1: Use the depreciation formula.

$$A = 15000(1 - 0.10)^5$$

$$A = 15000(0.9)^5$$

$$A = 15000 \times 0.59049$$

$$A = 8857.35$$

Thus, the car's value after 5 years is **$8857.35**.

Example 2: Appreciation Calculation
A house was purchased for \$120,000 and appreciates at a rate of 5% per year. Find its value after 8 years.
Solution:
Step 1: Use the appreciation formula.

$$A = 120000(1 + 0.05)^8$$

$$A = 120000(1.477455)$$

$$A = 177294.60$$

Thus, the house's value after 8 years is **\$177,294.60**.

Chapter 8: Miscellaneous Topics

8.1 Approximation and Estimation

Approximation and estimation are useful techniques in numerical computations to simplify calculations while maintaining accuracy.

Key Concepts:

- **Significant Figures:** Significant figures include all nonzero digits, any zeros between nonzero digits, and trailing zeros in a decimal number.
  - Examples of significant figures:
    - 123 has 3 significant figures.
    - 0.004567 has 4 significant figures (leading zeros are not significant).
    - 50.00 has 4 significant figures (trailing zeros in a decimal are significant).
  - Rounding to a given number of significant figures:
    - Identify the required number of significant figures.
    - Look at the next digit after the last significant figure:
      - If the next digit is 5 or greater, round up the last significant digit.
      - If the next digit is less than 5, leave the last significant digit unchanged.
  - Example: 0.00723456 rounded to 3 significant figures is **0.00723**.
  - Example: 456.78 rounded to 2 significant figures is **460** (since 6 is greater than 5, round up).
  - **Which zeros are significant?**
    - **Zeros between nonzero digits** (Captive Zeros) are significant.
      - Example: 105 has 3 significant figures.
      - Example: 20.08 has 4 significant figures.
    - **Leading zeros** (before the first nonzero digit) are not significant.
      - Example: 0.0047 has 2 significant figures.
      - Example: 0.000230 has 3 significant figures.
    - **Trailing zeros in a decimal number** are significant.
      - Example: 50.00 has 4 significant figures.
      - Example: 2.500 has 4 significant figures.
    - **Trailing zeros in a whole number without a decimal** are not significant.
      - Example: 1500 has 2 significant figures.
      - Example: 42000 has 2 significant figures.

- · However, 1500.0 has 5 significant figures (because of the decimal point).

- **Decimal Places:** The number of digits after the decimal point.
  - Example: 12.3456 rounded to 2 decimal places is 12.35.
- **Rounding:** Adjusting a number to a given decimal place or significant figure.
  - Example: 457 rounded to the nearest ten is 460.
- **Error Percentage:** The relative difference between an estimated value and the actual value.

$$\text{Percentage Error} = \left(\frac{|\text{Approximate Value} - \text{Exact Value}|}{\text{Exact Value}} \right) \times 100$$

- **Estimation Techniques:** Approximating calculations to simplify operations, such as using rounding to estimate sums and products.
- **Solving Equations Correct to n Decimal Places:** Finding numerical solutions to equations with a specified accuracy.

Examples:

Example 1: Rounding a Number to Significant Figures
Round 0.00723456 to 3 significant figures.
Solution:
The first three significant figures are 7, 2, and 3. Therefore, rounding to 3 significant figures:

$$0.00723456 \approx 0.00723.$$

Example 2: Calculating Percentage Error
An estimated value for a quantity is 48.3, while the actual value is 50. Find the percentage error.
Solution:
Step 1: Compute the absolute error.

$$|48.3 - 50| = 1.7.$$

Step 2: Compute the percentage error.

$$\text{Percentage Error} = \left(\frac{1.7}{50} \right) \times 100 = 3.4\%.$$

Thus, the percentage error is **3.4%**.

Example 3: Solving an Equation Correct to 2 Decimal Places
Solve $x^2 - 2x - 3 = 0$ correct to 2 decimal places.
Solution:
Using the quadratic formula:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2}$$

$$x = \frac{2 \pm 4}{2}.$$

Thus, the two roots are:

$$x = \frac{2+4}{2} = 3.00, \quad x = \frac{2-4}{2} = -1.00.$$

The solutions correct to **2 decimal places** are **$x = 3.00$ and $x = -1.00$**.

8.2 Symbolic Notation

Symbolic notation is used in logic and mathematics to express statements and relationships clearly and concisely.

Key Concepts:

- **Logical Statements:** A statement is a sentence that is either true or false but not both.
- **Truth Values:** A statement can be either:

$$\text{True (T)} \quad \text{or} \quad \text{False (F)}$$

- **Logical Connectives:**
  - **Negation ($\neg P$):** The opposite of a statement.

    $$\text{If } P \text{ is "It is raining," then } \neg P \text{ is "It is not raining."}$$

  - **Conjunction ($P \wedge Q$):** True if both statements are true.

    $$P \wedge Q \text{ is true only if both } P \text{ and } Q \text{ are true.}$$

  - **Disjunction ($P \vee Q$):** True if at least one statement is true.

    $$P \vee Q \text{ is true if either } P \text{ or } Q \text{ (or both) are true.}$$

  - Contrapositive: The contrapositive of an implication states that if $A \Rightarrow B$ is true, then its contrapositive $\neg B \Rightarrow \neg A$ is also true.

    $$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$$

    Example: If the statement "If it is raining, then the ground is wet" ($A \Rightarrow B$) is true, then the contrapositive "If the ground is not wet, then it is not raining" ($\neg B \Rightarrow \neg A$) must also be true.
  - **Implication ($P \Rightarrow Q$):** If P is true, then Q must be true.

    $$\text{"If it rains, then the ground is wet."}$$

  - Contrapositive Rule: If an implication $A \Rightarrow B$ is true, then its contrapositive $\neg B \Rightarrow \neg A$ is also true. This means that if "A implies B" is valid, then "Not B implies Not A" is also valid.

    $$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$$

    Example: If "If it is raining, then the ground is wet" ($A \Rightarrow B$) is true, then "If the ground is not wet, then it is not raining" ($\neg B \Rightarrow \neg A$) must also be true.

– **Biconditional ($P \Leftrightarrow Q$):** True if P and Q are either both true or both false.

"A shape is a square if and only if it has four equal sides and right angles."

Examples:

Example 1: Determine the truth value of the following statement:
"If 2 is an even number, then 3 is an odd number."
Solution: The statement can be written as:

$$P \Rightarrow Q, \quad \text{where } P : \text{"2 is even" and } Q : \text{"3 is odd"}.$$

Since both P and Q are true, the implication is **true**.

Example 2: Determine the truth value of the following compound statement: "It is raining and it is sunny."
Solution: The statement is of the form $P \wedge Q$. If it is raining (P is true) but it is not sunny (Q is false), then:

$$P \wedge Q = F.$$

Thus, the statement is **false**.

Example 2: Contrapositive of a Statement
Consider the statement: "If a number is divisible by 6, then it is divisible by 2."
Solution:
This can be written as an implication:

$$A \Rightarrow B, \quad \text{where } A : \text{"A number is divisible by 6" and } B : \text{"The number is divisible by 2"}.$$

The contrapositive of this statement is:

$$\neg B \Rightarrow \neg A, \quad \text{"If a number is not divisible by 2, then it is not divisible by 6."}$$

Since this contrapositive is logically equivalent to the original statement, it must also be **true**.

8.3 Sets of Numbers

Numbers can be categorized into different sets based on their properties. These sets form the foundation of number theory and algebra.

Key Concepts:

- **Natural Numbers ($\mathbb{N}$):** The set of positive counting numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

  Some definitions include 0 as a natural number: $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.

- **Whole Numbers ($\mathbb{W}$):** The set of natural numbers including zero:

$$\mathbb{W} = \{0, 1, 2, 3, \dots\}$$

- **Integers ($\mathbb{Z}$):** The set of whole numbers and their negative counterparts:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

- **Rational Numbers ($\mathbb{Q}$):** Numbers that can be expressed as a fraction $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

  Examples: $\frac{1}{2}, -3, 0.75, 5$.

- **Irrational Numbers:** Numbers that cannot be written as fractions, having non-repeating, non-terminating decimals.

$$\sqrt{2}, \pi, e$$

- **Real Numbers ($\mathbb{R}$):** The set of all rational and irrational numbers.

$$\mathbb{R} = \mathbb{Q} \cup \text{Irrational Numbers}$$

- **Complex Numbers ($\mathbb{C}$):** The set of numbers in the form:

$$a + bi, \quad \text{where } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}.$$

Examples:

Example 1: Classify the number $\frac{5}{2}$.
Solution: $\frac{5}{2}$ is a fraction of two integers, so it belongs to **rational numbers ($\mathbb{Q}$)**. Since it is not a whole number, it is not an integer.

Example 2: Determine whether $\sqrt{16}$ and $\sqrt{17}$ are rational.
Solution:

$$\sqrt{16} = 4, \quad \text{which is an integer and therefore rational.}$$

$$\sqrt{17} \approx 4.123, \quad \text{which is a non-repeating, non-terminating decimal, so it is irrational.}$$

Thus, $\sqrt{16}$ is **rational**, while $\sqrt{17}$ is **irrational**.